- The spectral Theorem:

  If A is symmetric then it has an orthonormal eigenbasis.

Idea about symmetry.

$$\boxed{x \cdot Ay \approx x^T Ay} = x^T A^T y = (Ax)^T y = Ax \cdot y$$

$\Downarrow$

Claim λ must be a real number... this isn't obvious, because λ is obtained by solving

$$\det\underbrace{(A - \lambda I) = 0}_{\text{polynomial equation...}}$$

Fundamental theorem of algebra implies that λ might be complex...

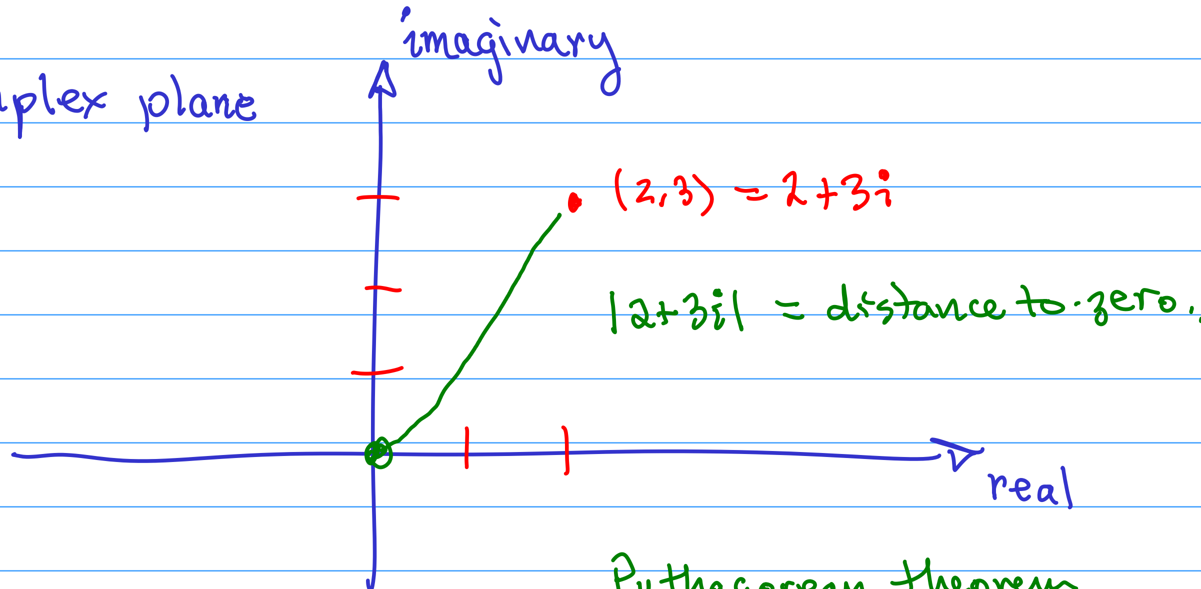
Note in λ is complex, then substituting λ into $\mathrm{Nul}(A - \lambda I)$ to find x is likely to obtain a complex vector as well...

Review of complex numbers...

$a+ib$ where $a,b \in \mathbb{R}$ and $i^2=-1$

Example $2+3i$

Complex plane



Pythagorean theorem

$$|2+3i| = \sqrt{2^2+3^2}$$

$$|2+3i|^2 = 2^2+3^2$$

Complex conjugate...

$$\overline{2+3i} = 2-3i$$

$$(2+3i)(2-3i) = 4 - \cancel{6i} + \cancel{6i} - (3i)^2 = 4+9$$

Therefore

$$|2+3i|^2 = (2+3i)\overline{(2+3i)}$$

This allows us to see the absolute value of a complex number in terms of a product...

Now even if x is a complex vector then the norm of x is real...

$$\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

$$\|x\|^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$$

$$= \underbrace{x_1 \bar{x}_1 + x_2 \bar{x}_2 + \cdots + x_n \bar{x}_n}_{\text{dot product}}$$

Thus

$\|x\|^2 = x \cdot \bar{x}$ and this is a real number...

Let $A \in \mathbb{R}^{n \times n}$ and $A = A^T$ and $Ax = \lambda x$.

Claim λ is real... Why?

$$Ax \cdot \bar{x} = \lambda x \cdot \bar{x} = \lambda \|x\|^2$$

$$\|$$

since A is real then $\bar{A} = A$

$$(Ax)^T \bar{x} = x^T A^T \bar{x} = x^T A \bar{x} = x^T \bar{A} \bar{x}$$

$$= x^T \overline{Ax} = x^T \overline{\lambda x} = \bar{\lambda} x^T \bar{x} = \bar{\lambda} \|x\|^2$$

Therefore $\lambda \|x\|^2 = \bar{\lambda} \|x\|^2$. Since $\|x\| \neq 0$ then $\lambda = \bar{\lambda}$ so λ is real...