Eigenvalues and eigenvectors that correspond to a symmetric matrix $A \in \mathbb{R}^{n\times n}$.

Showed that λ is real...

recall we used $\|x\|^2 = x \cdot \bar{x}$ for complex vectors and the fact that $A = A^T$ to pass A through the dot product to obtain $\lambda = \bar{\lambda}$ which means $\lambda \in \mathbb{R}$.

Show if λ_1 and λ_2 are two different eigenvalues then the corresponding eigenvectors are orthogonal.

Means $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$ and $\lambda_1 \neq \lambda_2$ implies $x_1 \cdot x_2 = 0$.

By the previous result we assume λ's and x's are real.

$$Ax_1 \cdot x_2 = (Ax_1)^T x_2 = x_1^T \overbrace{A^T x_2 = x_1^T A}^{\text{Symmetry}} x_2 = x_1 \cdot Ax_2$$

$$\begin{array}{ccc} Ax_1 \cdot x_2 & & x_1 \cdot Ax_2 \\ \| & & \| \\ \lambda_1 x_1 \cdot x_2 & & x_1 \cdot \lambda_2 x_2 \end{array}$$

Therefore $\lambda_1 x_1 \cdot x_2 = \lambda_2 x_1 \cdot x_2$

$(\lambda_1 - \lambda_2) x_1 \cdot x_2 = 0$

since $\lambda_1 \neq \lambda_2$ then $\lambda_1 - \lambda_2 \neq 0$

divide through since not zero

so $x_1 \cdot x_2 = 0$

If x_1 and x_2 weren't unit one could normalize them as $\frac{x_1}{\|x_1\|}$ and $\frac{x_2}{\|x_2\|}$

and now we have orthonormal eigenvectors...

Suppose you have two linearly independent eigenvectors with the same eigenvalue...

$$Ax_1 = \lambda x_1 \qquad Ax_2 = \lambda x_2$$

Use Gram-Schmidt to obtain q_1 and q_2

$$t_1 = x_1 \qquad q_1 = \frac{t_1}{\|t_1\|}$$

$$t_2 = x_2 - (q_1 \cdot x_2) q_1 \qquad q_2 = \frac{t_2}{\|t_2\|}$$

Then q_1 and q_2 are orthonormal eigenvectors.