Diagonalization of Symmetric Matrices...

Let $A \in \mathbb{R}^{n \times n}$ with $A = A^T$. By the spectral theorem there is an eigenbasis for A.

Thus we have n eigenvector x_n such that

$$Ax_i = \lambda_i x_i \quad \text{and} \quad x_i \cdot x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Note also that the λ's are all real numbers.

Any matrix can be diagonalized by a similarity transformation S whose columns are the eigenvectors proved there's an eigenbasis...

$$AS = SD \quad \text{or} \quad A = SDS^{-1}$$

matrix of eigenvectors

diagonal matrix of eigenvalues

In the case A is symmetric

with this choice

① It's guaranteed there is a basis of eigenvectors

② They can be choosen to be orthonormal.

$\Downarrow$

S is a square matrix with orthonomal columns...

This means $S^TS = I$ and so $S^{-1} = S^T$.

Finish 7.1 with...

$$A = SDS^T \quad \text{when } A = A^T.$$

Example:

16. $\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} = A$

Find eigenvalues...

$$\det(A - \lambda I) = \det\begin{bmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{bmatrix} = (6-\lambda)(9-\lambda) - 4 = 0$$

always a monic polynomial

monic polynomial since there is no coefficient on λ^2

$$54 - 6\lambda - 9\lambda + \lambda^2 - 4 = \lambda^2 - 15\lambda + 50 = 0$$

rational root theorem $5 \cdot 5 \cdot 2 = 50$

$\pm 1, \pm 2, \pm 5, \pm 10, \pm 25, \pm 50$

Since $5^2 - 15 \cdot 5 + 50 = 0$ then $\lambda - 5$ is a factor

$$(\lambda - 5)(\lambda - 10) = 0$$

Therefore $\lambda = 5$ and $\lambda = 10$ are the eigenvalues...

$\lambda = 5$

$$\text{Nul}\left(\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{Nul}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

eigenvector satisfies $1u_1 - 2u_2 = 0$

$$u_1 = 2u_2$$
$$u_2 = \text{free}$$

$$u = \begin{bmatrix} 2u_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} u_2$$

Want to be a unit vector

Eigen vector corresponding to $\lambda_1 = 5$

is $x_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 10$

$$\text{Nul}\left(\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} - 10 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \text{Nul} \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$$

Finding Nulspace

$$-2u_1 - u_2 = 0$$

$$\begin{cases} u_2 = -2u_1 \\ u_1 = \text{free} \end{cases}$$

$$u = \begin{bmatrix} 1 \\ -2 \end{bmatrix} u_1$$

Eigenvector corresponding to $\lambda_2 = 10$

is $x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Finally, use the eigenbasis to diagonalise the matrix A.

$$S = \left[x_1 \,\middle|\, x_2 \right] = \left[\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \,\middle|\, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \right] = \frac{1}{\sqrt{5}} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}}$$

orthogonal matrix.

note S is not always symmetric that was just lucky...

Check that

$$S^T S = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

Note another choice for x_2 could have been

$$x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Then

$$S = \left[x_1 \,\middle|\, x_2 \right] = \frac{1}{\sqrt{5}} \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}$$

not so lucky this time it was not symmetric...

check

$$S^T S = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = I$$

In summary

$$\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \end{array} \begin{bmatrix} \overset{\lambda_1}{5} & 0 \\ 0 & \overset{\lambda_2}{10} \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Application of the Spectral Theorem...

Singular value decomposition...

Given any $A \in \mathbb{R}^{n \times n}$ that's not symmetric

then $B = A^T A$ is symmetric

and so is $C = A A^T$.