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The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

Let $v_1, v_2, \dots, v_p \in H$

Lin ind means $A = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}}$ $\in \mathbb{R}^{n \times p}$

So there are vectors $w_1, w_2, \dots, w_p$ that form a basis for H. That is

① $w_1, w_2, \dots, w_p$ span H

② $w_1, w_2, \dots, w_p$ lin. independent

$$M = \begin{bmatrix} w_1 & w_2 & \cdots & w_p \end{bmatrix} \in \mathbb{R}^{n \times p}$$

Since $v_1, v_2, \dots, v_p \in H$ then

$$\boxed{v_1 = My_1, \quad v_2 = My_2, \dots, \quad v_p = My_p}$$

for some vectors $y_1, y_2, \dots, y_p \in \mathbb{R}^p$

p vectors y_i of length p.

$$C = \begin{bmatrix} y_1 & y_2 & \cdots & y_p \end{bmatrix} \in \mathbb{R}^{p \times p}$$

Claim C is invertible... (need to show C has no free variables...)

squareness of the C matrix

& A has lin. independent columns...

Note $A = MC$

for C to have no free variables means

$Cz = 0$ has only one solution $z = 0$.

For contradiction, suppose C had free variables...

that means there is a vector $z \in \mathbb{R}^p$, $z \neq 0$

such that $Cz = 0$.

Then $Az = MCz = M0 = 0$

This means that A has free variables... since $z \neq 0$

lin ind of cols of A means The only solution of $Az = 0$ is $z = 0$

That's a contradiction

Therefore C has no free variables

Thus C is invertible...

Back to showing the vectors $v_1, \dots, v_p$ are a basis, that is, they span the subspace H.

Need to show for every $b \in H$ there is $x \in \mathbb{R}^p$ such that $b = Ax$

Since the columns in M span H then there is $y \in \mathbb{R}^p$ such that $b = My$... (by definition of basis)

Note since M is a basis then $b \in H$ implies there is $y \in \mathbb{R}^p$ such that $b = My$...

$$A = MC$$

want this
$$Az = b$$

$$b = My$$

$$Az = MCz$$

$$y = Cz$$

$$z = C^{-1}y$$

Then $Az = AC^{-1}y = M\cancel{C}\cancel{C^{-1}}y = My = b$