REM 15

The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

rows

Hypothesis: $\dim H = p$ means there is a basis with p vectors...

Let $w_1, w_2, \dots, w_p \in \mathbb{R}^n$ be a basis of H.

① $w_1, \dots, w_p$ are indep.
② $w_1, \dots, w_p$ span H

Let $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ be a linearly independent set in H.

to show $v_1, \dots, v_p$ form a basis of H, I need to show they span H.

Since $v_1, \dots, v_p \in H$ then each can be expressed in terms of the basis.

Let $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix} \in \mathbb{R}^{n \times p}$ and $M = \begin{bmatrix} w_1 & w_2 & \cdots & w_p \end{bmatrix} \in \mathbb{R}^{n \times p}$

basis

Since w's are a basis they span H. Thus $v_1 \in H$ so it can be written as a span of the w's. Thus

these equations can be written

$$\left.\begin{aligned} v_1 &= My_1 \text{ for some } y_1 \in \mathbb{R}^p \\ &\vdots \\ v_p &= My_p \text{ for some } y_p \in \mathbb{R}^p \end{aligned}\right\} \quad A = MC$$

rows

$$C = \begin{bmatrix} y_1 & y_2 & \cdots & y_p \end{bmatrix} \in \mathbb{R}^{p \times p}$$

columns

C is invertible if there are no free variables in the row eschelon form of C.

Claim: In fact C is invertible...

For contradiction, suppose C were not invertible... then there would be free variables... $Cz = 0$ has a solution $z \neq 0$.

Then $Az = MCz = M0 = 0$ for $z \neq 0$.

But since the columns of A are linearly independent the only solution to $Az = 0$ is $z = 0$... Contradiction!

Therefore, it must be that C is invertible

I need to show they span H.

Let $b \in H$ need to find $x \in \mathbb{R}^p$ such that $Ax = b$...

Since M is made from a basis... there is y such that $My = b$.

Recall $A = MC$

want $MCx = b$

$Cx = y$

since invertible $x = C^{-1}y$

Check:

$$Ax = AC^{-1}y = M\cancel{C}\cancel{C^{-1}}y = My = b$$