Idea: Compute the determinant of any $n \times n$ matrix efficiently... by factoring A into simpler parts... and the combining thing...

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$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \quad \text{for some fixed } i$$

Suppose $n=4$ and $A \in \mathbb{R}^{4\times 4}$

to compute $\det A$ using the definition involves
computing 4 determinants of size 3×3

to compute a 3×3 determinant using the definition involves

computing 3 determinants of size 2×2

and each of those by
computing 2 determinants of size 1×1

Total number of terms in the sum $4\cdot 3\cdot 2 = 4!$

$$A = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \in \mathbb{R}^{5\times 5}$$

Suppose can make LU factorization...

$A = LU$ ← upper triangular...

where L ← lower triangular with 1's on the diagonal

Recall: U is the row echelon form of A made using Gaussian elimination...

L is containing the multipliers used in the elimination steps

$r_i \leftarrow r_i - \alpha_{ij} r_j$

since making zeros below the pivots then $i > j$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \alpha_{21} & 1 & 0 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 1 & 0 & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 & 0 \\ \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & 1 \end{bmatrix}$$

Note since L and U are triangular it easy to compute their determinants.

$$\det L = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

$$U = \begin{bmatrix} 2 & u_{12} & u_{13} & u_{14} & u_{15} \\ 0 & 4 & u_{23} & u_{24} & u_{25} \\ 0 & 0 & 6 & u_{34} & u_{35} \\ 0 & 0 & 0 & -8 & u_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ ← row eschelon form

$$\begin{array}{r} 8 \\ 6 \\ \hline 48 \\ 8 \\ \hline 384 \end{array}$$

$\det U = 2 \cdot 4 \cdot 6(-8)1 = -384$

Count zeros = # of row operation to make the zeros ...

$4+3+2+1$ row operation ... each row is n terms

$(4+3+2+1) \cdot 5 = 50$ terms ...

about $2n^2$

$$\begin{array}{r} + \; 4 \\ \hline 54 \end{array}$$ to mult the diagonal

$$\begin{array}{r} 20 \\ 6 \\ \hline 120 \end{array}$$

$5! = \underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{n!} = 120$ terms ...

To combine things $\det A = (\det L)(\det U) = -384$.

understand this ...

3.2: Properties of determinants...

$(i \neq j)$

$r_i \leftarrow r_i - \alpha r_j$

$r_i \leftrightarrow r_j$

$r_i \leftarrow \alpha r_i$

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Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Examples 2x2 case

(a) $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1\cdot 4 - 2\cdot 3 = -2$

$r_2 \leftarrow r_2 - 3r_1$

$\det \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = 1\cdot(-2) - 2\cdot 0 = -2$

(b) $r_1 \leftrightarrow r_2$

$\det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = 3\cdot 2 - 4\cdot 1 = 2$

(c) $\det\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = 1\cdot4-2\cdot3 = -2$

$5(-2) = -10$

$r_2 \leftarrow 5r_2$

$\det\begin{bmatrix}1 & 2\\ 15 & 20\end{bmatrix} = 1\cdot20 - 2\cdot15 = -10$

Note if I mult. the entire matrix by 5 something else happens

(c)

$5\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}5 & 10\\ 15 & 20\end{bmatrix}$ $\quad r_1 \leftarrow 5r_1$, $r_2 \leftarrow 5r_2$

$\det\begin{bmatrix}5 & 10\\ 15 & 20\end{bmatrix} = 5\cdot5\cdot\det\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

$= (25)(-2) = -50$