Section 3.3 Cramer's rule for solving $Ax=b$ using determinants.

Example: $n=4$ Solve $\boxed{Ax=b}$ for $A \in \mathbb{R}^{4\times 4}$

invertible... means $\det A \neq 0$.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

replace one of the column with x

Try the third column ..

$$I_3 = \begin{bmatrix} 1 & 0 & x_1 & 0 \\ 0 & 1 & x_2 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & x_4 & 1 \end{bmatrix}$$

what is $\det I_3 = ?$

Case $x_3 \neq 0$. Then $r_4 \leftarrow r_4 - \frac{x_4}{x_3} r_3$

$$\det I_3 = \det \begin{bmatrix} 1 & 0 & x_1 & 0 \\ 0 & 1 & x_2 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = x_3$$

Want to use $\det(AB) = \det(A)\det(B)$

$$AI_3 = A\begin{bmatrix} 1 & 0 & x_1 & 0 \\ 0 & 1 & x_2 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & x_4 & 1 \end{bmatrix} = \left[\, A\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \;\middle|\; A\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \;\middle|\; Ax \;\middle|\; A\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \,\right]$$

$Ax = b$

Third column

$$= \begin{bmatrix} a_{11} & a_{12} & b_1 & a_{14} \\ a_{21} & a_{22} & b_2 & a_{24} \\ a_{31} & a_{32} & b_3 & a_{34} \\ a_{41} & a_{42} & b_4 & a_{44} \end{bmatrix} = A_3$$

This is the matrix A where the third column has been replaced by b.

Thus,

$$\det AI_3 = \det A_3$$

$$\det A \; \boxed{\det I_3} = \det A_3$$

$$(\det A)\, x_3 = \det A_3$$

$$x_3 = \frac{\det A_3}{\det A}$$

Same pattern gives Cramer's rule ...

$$x_j = \frac{\det A_j}{\det A} \quad \text{for } j = 1, \dots, n$$

where A_j is the matrix A with the j^{th} column replace by b

REM 7

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

(ith column here)

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \dots, n \tag{1}$$

How to find the inverse matrix?

Let $f(x) = Ax$ and f^{-1} be the inverse of f.

Then the matrix for f^{-1} is given by

$$M = \left[f^{-1}(e_1) \,\middle|\, f^{-1}(e_2) \,\middle|\, \dots \,\middle|\, f^{-1}(e_n) \right]$$

and $f^{-1}(x) = Mx$

& write the M obtained in this way by A^{-1}

If I use Cramers rule to find $\boxed{f^{-1}(e_k)}$ for $k = 1, \dots, n$ then I get an explicit formula for the inverse...

$z = f^{-1}(e_k)$ means $f(z) = e_k$

$Az = e_k$

& solve for z using Cramer's rule...

Thus

$$z_i = \frac{\det(A_i(e_k))}{\det A}$$

Put everything together and get

swapping order of indices is transpose...

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

adjugate matrix

where $C_{ji} = (-1)^{i+j} \underbrace{\det A_{ji}}_{\text{determinants of } (n-1)\times(n-1) \text{ matrices}}$

From the definition of determinant

$= [a_{ij}]$, the **(i, j)-cofactor** of A

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

The matrix of cofactors on the right side of (4) is called the **adjugate** (or **classical adjoint**) of A, denoted by adj A. (The term *adjoint* also has another meaning in advanced texts on linear transformations.) The next theorem simply restates (4).

$\boxed{\text{just the transpose}}$