Section 3.3 Solving $Ax=b$ using Cramer's rule

assume A is invertible...

<u>Example:</u> $n=4$ $\qquad Ax=b$ where $x\in\mathbb{R}^4$, $A\in\mathbb{R}^{4\times 4}$ and $b\in\mathbb{R}^4$.

$\begin{cases} \det A \neq 0 \text{ implies } A \text{ is invertible} \\ A \text{ invertible implies } \det A \neq 0. \end{cases}$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

replace a column in I with the vector x.

(second column replaced by x)

$$I_2 = \begin{bmatrix} 1 & x_1 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & x_3 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{bmatrix}$$

what is $\det I_2 = ?$

pivot

Try doing elimination steps to make this upper triangular...

Case $x_2 \neq 0$ $\qquad$ case $x_2 = 0$

Since $x_2 \neq 0$ can use it as a pivot

$$r_3 \leftarrow r_3 - \frac{x_3}{x_2} r_2$$
$$r_4 \leftarrow r_4 - \frac{x_4}{x_2} r_2$$

eliminate

$$\begin{bmatrix} 1 & x_1 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & x_3 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & x_1 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

new matrix

product on diagonal..

$$\det I_2 = x_2$$

If $x_2 = 0$ then

expand on this row

$$I_2 = \begin{bmatrix} 1 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x_3 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{bmatrix}$$

$$\det I_2 = 0 = x_2$$

in both cases $\boxed{\det I_2 = x_2}$

From Friday we had $\det(AB) = \det(A)\det(B)$

set $B = I_2$

$$AI_2 = A \begin{bmatrix} 1 & x_1 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & x_3 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{bmatrix} = \left[A\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \middle| Ax \middle| A\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \middle| A\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right]$$

(by the definition of matrix matrix multiplication)

$$Ax = b$$

$$= \begin{bmatrix} a_{11} & b_1 & a_{13} & a_{14} \\ a_{21} & b_2 & a_{23} & a_{24} \\ a_{31} & b_3 & a_{33} & a_{34} \\ a_{41} & b_4 & a_{43} & a_{44} \end{bmatrix} = A_2$$

A with the 2nd column replaced by b.

$$A\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}a_{13}\\ \\ \\ \\ \end{bmatrix}$$

Therefore $AI_2 = A_2$ ← A with second column replaced by b.

↑ I with second column replaced by x

$$\det I_2 = x_2$$

$$\det(AI_2) = \det A_2$$

$$\det(A)\det(I_2) = \det A_2$$

$$(\det A)\, x_2 = \det A_2$$

$$x_2 = \frac{\det A_2}{\det A}$$ ← (A with second column replaced by b.)

Generalize

(A with i^{th} column replaced by b.)

$$\boxed{x_i = \frac{\det A_i}{\det A} \quad \text{for } i = 1, \ldots, n}$$

Solution to $Ax=b$

↑ Cramer's rule

Formula for the inverse A^{-1}.

In the past the inverse matrix was the matrix corresponding to the inverse function f^{-1} where $f(x) = Ax$...

$$A^{-1} = \left[\underbrace{f^{-1}(e_1)}_{z} \;\middle|\; f^{-1}(e_2) \;\middle|\; \cdots \;\middle|\; f^{-1}(e_n) \right]$$

find this means solve

$A z = e_1$

Solve for z using Cramer's rule and substitute in the definition of A^{-1}

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$$

ofactor of A. By (2), the (i, j)-entry of A^{-1} is the cofactor C_{ji}
that the subscripts on C_{ji} are the reverse of (i, j).] Thus

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

↓ count in columns