Chapter 4 is an expansion of the ideas in chapters 1 and 2.

we'll skip it for now but

- Please read it on your own
- Let me know how that goes
- I'll jump back as needed for anything in chapter 5, 6 or 7.

$A = QR$ Factorization

Q has columns which are orthonormal vectors...

vectors are a right angles to each other and of unit length

$A \in \mathbb{R}^{m \times n}$ $\quad$ $Q \in \mathbb{R}^{m \times n}$ $\quad$ $R \in \mathbb{R}^{n \times n}$

R: invertible upper triangular matrix

since $A = QR$ and R is invertible

then $\operatorname{Col} A = \operatorname{Col} Q$

note, since orthonormal means independent, then if this equality hold it must be that the columns of A are linearly independent

otherwise (not)

$\left.\begin{array}{l}\dim \operatorname{Col} A \\ \dim \operatorname{Col} Q\end{array}\right\}$ have to be the same

Need an $\overbrace{\text{algorithm}}^{\text{method, idea}}$ to make a factorization...

For example $A = LU$ was obtained by Gaussian elimination...

$U \leftarrow$ row eschelon form obtained by elimination steps...

$$r_i \leftarrow r_i - \alpha_{ij} r_j \qquad i > j$$

$$L = \begin{bmatrix} 1 & & & & \\ \alpha_{21} & 1 & & 0 & \\ \alpha_{31} & \alpha_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & 1 \end{bmatrix}$$

Example

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad r_2 \leftarrow r_2 - \frac{4}{3} r_1$$

$$\alpha_{21} = \frac{4}{3}$$

Algorithm for finding QR is called Gram-Schmidt orthogonalization

$$A = \left[\, a_1 \,\middle|\, a_2 \,\middle|\, \cdots \,\middle|\, a_n \,\right] \in \mathbb{R}^{m \times n}$$

Assume the a_i's are linearly independent...

algorithm

$$t_1 = a_1 \qquad\qquad q_1 = \frac{t_1}{\boxed{\|t_1\|}}$$

← unit vector of t_1

$$t_2 = a_2 - \boxed{(a_2 \cdot q_1)}\, q_1$$

claim t_2 is perpendicular to q_1

$$t_2 \cdot q_1 = \left(a_2 - (a_2 \cdot q_1)\, q_1\right) \cdot q_1$$

$$= a_2 \cdot q_1 - (a_2 \cdot q_1)(q_1 \cdot q_1)$$

Then $q_1 \cdot q_1 = 1$

$$= a_2 \cdot q_1 - (a_2 \cdot q_1) = 0 \qquad\qquad q_2 = \frac{t_2}{\|t_2\|}$$

$$t_3 = a_3 - (a_3 \cdot q_1)\, q_1 - (a_3 \cdot q_2)\, q_2 \qquad\qquad q_3 = \frac{t_3}{\|t_3\|}$$

$$\vdots$$

$$t_n = a_n - (a_n \cdot q_1)\, q_1 - \cdots - (a_n \cdot q_{n-1})\, q_{n-1} \qquad\qquad q_n = \frac{t_n}{\|t_n\|}$$

$$Q = \left[\, q_1 \,\middle|\, q_2 \,\middle|\, \cdots \,\middle|\, q_n \,\right] \qquad R = \begin{bmatrix} \|t_1\| & & & \\ & \|t_2\| & & \\ & & \ddots & \\ & & & \|t_n\| \end{bmatrix}$$