Finish example from last time...

$q_1 = \frac{1}{2\sqrt{5}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$ $\quad q_2 = \frac{1}{\sqrt{20}} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$ $\quad q_3 = \frac{1}{\sqrt{20}} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$

$$Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \frac{1}{\sqrt{20}} \begin{bmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix}$$

$$R = \begin{bmatrix} \|t_1\| & q_1 \cdot a_2 & q_1 \cdot a_3 \\ 0 & \|t_2\| & q_2 \cdot a_3 \\ 0 & 0 & \|t_3\| \end{bmatrix} = \begin{bmatrix} \sqrt{20} & -40/\sqrt{20} & 30/\sqrt{20} \\ 0 & \sqrt{20} & -10/\sqrt{20} \\ 0 & 0 & \sqrt{20} \end{bmatrix}$$

$$\|t_1\| = \sqrt{3^2+1^2+1^2+3^2} = \sqrt{20} = 2\sqrt{5}$$

$\|t_2\| = \sqrt{20}$ $\quad \|t_3\| = \sqrt{20}$

$$q_1 \cdot q_2 = \frac{-40}{\sqrt{20}}$$

dot product

$$\begin{array}{r} -15 \\ 1 \\ -5 \\ -21 \\ \hline -40 \end{array}$$

$$\Bigg(\underbrace{\begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}}_{q_2} \cdot \underbrace{\frac{1}{\sqrt{20}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}}_{q_1} \Bigg)$$

$q_1 \cdot a_3$

$$\left(\begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \cdot \frac{1}{2\sqrt{5}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \right) = \frac{1}{\sqrt{20}} 30 \checkmark$$

dot products

$$\begin{array}{r} 3 \\ 1 \\ 2 \\ 24 \\ \hline 30 \end{array}$$

$q_2 \cdot a_3$

$$\left(\begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \cdot \frac{1}{\sqrt{20}} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} \right) = \frac{1}{\sqrt{20}} (-10) \checkmark$$

$$\begin{array}{r} 1 \\ 3 \\ -6 \\ -8 \\ \hline -10 \end{array}$$

$$Q = \left[q_1 \,\middle|\, q_2 \,\middle|\, q_3 \right] = \frac{1}{\sqrt{20}} \begin{bmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix}$$

$$R = \begin{bmatrix} \|t_1\| & q_1 \cdot a_2 & q_1 \cdot a_3 \\ 0 & \|t_2\| & q_2 \cdot a_3 \\ 0 & 0 & \|t_3\| \end{bmatrix} = \begin{bmatrix} \sqrt{20} & -40/\sqrt{20} & 30/\sqrt{20} \\ 0 & \sqrt{20} & -10/\sqrt{20} \\ 0 & 0 & \sqrt{20} \end{bmatrix}$$

$$\sqrt{20} = \frac{20}{\sqrt{20}}$$

$$R = \frac{10}{\sqrt{20}} \begin{bmatrix} 2 & -4 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$Q = \frac{1}{\sqrt{20}} \begin{bmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix} \quad \frac{1}{2} \begin{bmatrix} 6 & -10 & 2 \\ 2 & 2 & 2 \\ -2 & 10 & -4 \\ 6 & -14 & 16 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

Thus $A = QR$. Now what?

$$Ax = b$$

Note

$$Q = \left[q_1 \,\middle|\, q_2 \,\middle|\, q_3 \right] \in \mathbb{R}^{4\times 3}$$

where q_i's are orthonormal...

$$q_i \cdot q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\underbrace{Q^T}_{3\times 4}\underbrace{Q}_{4\times 3} = \begin{bmatrix} q_1^T \\ \hline q_2^T \\ \hline q_3^T \end{bmatrix} \left[q_1 \,\middle|\, q_2 \,\middle|\, q_3 \right] = \begin{bmatrix} q_1\cdot q_1 & q_1\cdot q_2 & q_1\cdot q_3 \\ q_2\cdot q_1 & q_2\cdot q_2 & q_2\cdot q_3 \\ q_3\cdot q_1 & q_3\cdot q_2 & q_3\cdot q_3 \end{bmatrix}$$

$\in \mathbb{R}^{3\times 3}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since $A \in \mathbb{R}^{4\times 3}$ then there could not be a pivot in each column, thus not all values of b lead to a consistent system...

Given $\boxed{Ax = b}$

$QRx = b$ (mult both sides by Q^T)

$Q^TQRx = Q^Tb$

$IRx = Q^Tb$

$\boxed{Rx = Q^Tb}$ can always solve this equation...

using QR factorization obtained this

note also, since the diagonal of R is the $\|\varepsilon_i\|$'s then R is invertible...

Since R is upper triangular, can solve for x without any elimination steps only back-substitution is needed.

Will actually solve $Ax = b$ for some different values of b next time...