$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

$$Q = \frac{1}{\sqrt{20}} \begin{bmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix} \qquad R = \frac{10}{\sqrt{20}} \begin{bmatrix} 2 & -4 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Solve $Ax = b$... where $A = QR$

Since R is an invertible matrix then $\mathrm{Col}\,A = \mathrm{col}\,Q$

Suppose $\underset{m\times n}{M} = \underset{m\times k}{C}\,\underset{k\times n}{B}$ then $\mathrm{col}\,M \subseteq \mathrm{col}\,C$.

$\mathrm{Col}\,C = \{ Cz : z \in \mathbb{R}^k \}$

$\boxed{\mathrm{Col}\,B = \{ Bx : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^k}$

note $z \in \mathrm{Col}\,B$ mean that $z = Bx$ for some $x \in \mathbb{R}^n$

$\mathrm{Col}\,M = \{ Mx : x \in \mathbb{R}^n \} = \{ C(Bx) : x \in \mathbb{R}^n \}$

$z = Bx$

same thing

$= \{ Cz : z = Bx \text{ and } x \in \mathbb{R}^n \} = \{ Cz : z \in \mathrm{Col}\,B \}$

$\subseteq \{ Cz : z \in \mathbb{R}^k \}$ since $\mathrm{Col}\,B \subseteq \mathbb{R}^k$ Thus $\mathrm{Col}\,M \subseteq \mathrm{Col}\,C$.

Back to this...

Solve $Ax = b$... where $A = QR$

Since R is an invertible matrix then $\text{Col}\,A = \text{col}\,Q$

By the previous page $\text{Col}\,A \subseteq \text{Col}\,Q$...

Invertibility:

$$A = QR$$
$$AR^{-1} = Q(RR^{-1})$$
$$AR^{-1} = QI$$
$$Q = AR^{-1}$$

By the previous page applied to $\text{Col}\,Q \subseteq \text{Col}\,A$

combine these

$$\text{Col}\,A = \text{Col}\,Q$$

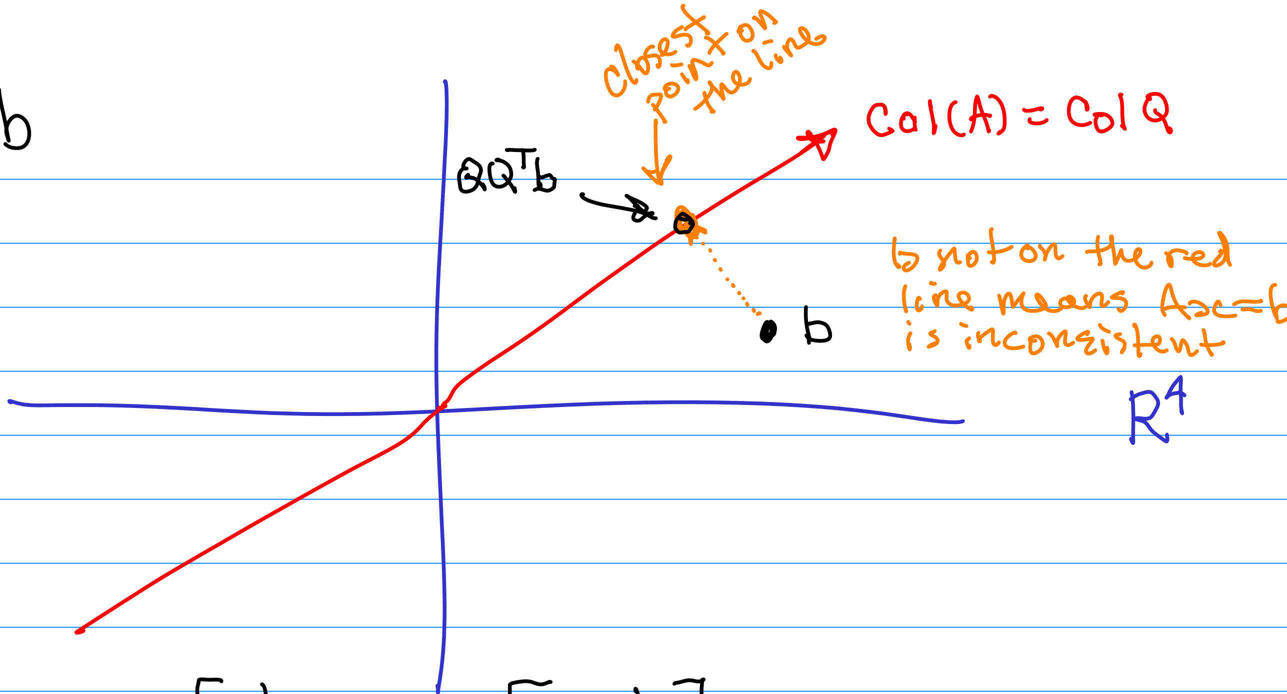
The column space of A is confusing, but the column space of Q is easy to understand because the columns of Q form an orthonormal basis of $\text{Col}\,Q$.

Now we can interpret the equation.

$$Rx = Q^T b$$

$$Q^T b = \begin{bmatrix} q_1^T \\ \hline q_2^T \\ \hline q_3^T \end{bmatrix} b = \begin{bmatrix} q_1 \cdot b \\ q_2 \cdot b \\ q_3 \cdot b \end{bmatrix}$$

$Ax = b$



$$QQ^Tb = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1 \cdot b \\ q_2 \cdot b \\ q_3 \cdot b \end{bmatrix} = (q_1 \cdot b)q_1 + (q_2 \cdot b)q_2 + (q_3 \cdot b)q_3$$