Last time we had $A \in \mathbb{R}^{m\times n}$ where $m \geq n$ and $A$ had linearly independent columns...

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} \in \mathbb{R}^{4\times 2}$$

$$\mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$

factor $\underset{m\times n}{A} = \underset{m\times n}{Q}\,\underset{n\times n}{R}$ ← upper triangular

matrix with orthonormal columns

Using Gram-Schmidt algorithm...

Solved $Rx = Q^Tb$

interpret this as the error in the equation $Ax=b$

Claim: The solution minimizes $\|Ax-b\|$ ...

Explain why...

$QQ^T$ is a projection

$QQ^Tb = Ax$

$\operatorname{Col} A = \operatorname{Col} Q$


Geometrically since the norm corresponds to distances computed using the Pythagorean theorem, the minimum is give by orthogonality...

Need to show

$\boxed{\checkmark}$ $(Ax-b)\cdot y = 0$ for every $y \in \operatorname{Col} A$ ...

- ☑ First solve for $x$ in $Rx = Q^Tb$
- ☑ Second plug that $x$ into $Ax$

Since $R$ is invertible then $R^{-1}$ exist's

$$R^{-1}Rx = R^{-1}Q^Tb$$

$$x = R^{-1}Q^Tb \quad ☑$$

plug in

recall $A = QR$

plug it in

$$Ax = A R^{-1}Q^Tb = Q\cancel{R}\cancel{R^{-1}}Q^Tb = QQ^Tb \quad ☑$$

Need to show

$(QQ^Tb - b)\cdot y = 0$ for every $y \in \operatorname{Col} Q$

Since $\operatorname{Col} Q$ is $\operatorname{Span}\{q_1, q_2, \ldots, q_n\}$

Need to show

$(QQ^Tb - b)\cdot q_k = 0$ for every $q_k$

☑

$$QQ^Tb = \begin{bmatrix} q_1 | q_2 | \cdots | q_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \hline q_2^T \\ \hline \vdots \\ \hline q_n^T \end{bmatrix} b = q_1(q_1 \cdot b) + q_2(q_2 \cdot b) + \cdots + q_n(q_n \cdot b)$$

$q_i$'s are orthonormal

$$q_i \cdot q_k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$$QQ^Tb \cdot q_k = \big(q_1(q_1 \cdot b) + q_2(q_2 \cdot b) + \cdots + q_n(q_n \cdot b)\big) \cdot q_k = \cancel{(q_k \cdot q_k)}\,(q_k \cdot b)$$

$$= q_k \cdot b \quad \square$$

$b \cdot q_k = q_k \cdot b$ the same

Therefore

$$(QQ^Tb - b) \cdot q_k = q_k \cdot b - q_k \cdot b = 0 \qquad \text{for every } q_k.$$

This orthogonality of $Ax - b$ with $\operatorname{col} A$ implies that $\|Ax - b\|$ is minimal when $x$ is the solution to $Rx = Q^Tb$.

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Connect solving $Rx = Q^Tb$ to solving the normal equations $\boxed{A^TAx = A^Tb}$

Recall $A = QR$. substitute

Since $Q$ has orthonormal columns the $Q^TQ = I$.

$$(QR)^T(QR)x = (QR)^Tb$$

$$R^T \cancel{Q^T} \cancel{Q} R x = R^T Q^T b$$

$$\cancel{R^T} R x = \cancel{R^T} Q^T b$$

$$\cancel{(R^T)^{-1}} \cancel{R^T} R x = \cancel{(R^T)^{-1}} \cancel{R^T} Q^T b$$

same equations...

$$\boxed{Rx = Q^T b}$$

all that's left is to cancel the $R^T$ on each side...

To do this without changing the solutions $x$ I need to know that $R^T$ is invertible...

Is it? YES.

**Reason 1**: Since $R$ is upper triangular and invertible than the diagonal entries of $R$ are all non-zero.

It follows that $R^T$ is lower triangular and again the diagonal entrier of $R^T$ are all non-zero. Therefore $R^T$ is invertible...

**Reason 2**: Given any matrix $M$ which is invertible then $M^T$ is invertible... Why...

$B = (M^{-1})^T$ exists because $M$ is invertible

$$M^T B = M^T (M^{-1})^T = (M^{-1} M)^T = I^T = I$$

Since $M^T$ and $B$ are square matrices this means that $B = (M^T)^{-1}$. In fact $(M^{-1})^T = (M^T)^{-1}$.

This page was added after class ——

- Check that solving $A^TAx = A^Tb$ gives the same answer as $Rx = Q^Tb$ for the problem worked in class last monday...

$\in \mathbb{R}$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \qquad , \mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 34 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 17 \\ 48 \end{bmatrix}$$

Thus $\begin{bmatrix} 4 & 6 \\ 6 & 34 \end{bmatrix} x = \begin{bmatrix} 17 \\ 48 \end{bmatrix}$

The answer from last time was

are the same

$$x = \begin{bmatrix} 29/10 \\ 9/10 \end{bmatrix} \text{ minimizes } \|Ax - b\|.$$

Plug in this to check whether that's also a solution to $A^TAx = A^Tb$.

$$\begin{bmatrix} 4 & 6 \\ 6 & 34 \end{bmatrix} \begin{bmatrix} 29/10 \\ 9/10 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 170 \\ 480 \end{bmatrix} = \begin{bmatrix} 17 \\ 48 \end{bmatrix}$$

Therefore both $A^TAx = A^Tb$ and $Rx = Q^Tb$ have the same solutions...