Matrix vector mult. and Matrix-Matrix mult

linear eq:

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 7 \\ -1x_1 + 2x_2 - 3x_3 = 4 \\ 5x_1 - 1x_2 + x_3 = 1 \end{cases}$$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 - 2x_3 \\ -1x_1 + 2x_2 - 3x_3 \\ 5x_1 - 1x_2 + x_3 \end{bmatrix}$$

Then the system can be written

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} \leftarrow b$$

The function f is called a linear function.

Definition:

① $f(a+b) = f(a) + f(b)$

② $f(\alpha a) = \alpha f(a)$ for α is scalar and a, b vectors.

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 - 2x_3 \\ -1x_1 + 2x_2 - 3x_3 \\ 5x_1 - 1x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 2 & -3 \\ 5 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

[This is the definition of Matrix-vector multiplication]

What structure is present in this definition:

(2) Column Structure:

$$\begin{bmatrix} 2 & 3 & -2 \\ -1 & 2 & -3 \\ 5 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} x_3$$

(1) Row structure

$$\begin{bmatrix} 2 & 3 & -2 \\ -1 & 2 & -3 \\ 5 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (2,3,-2) \cdot (x_1,x_2,x_3) \\ (-1,2,-3) \cdot (x_1,x_2,x_3) \\ (5,-1,1) \cdot (x_1,x_2,x_3) \end{bmatrix}$$

Therefore $f(x) = Ax$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 2 & -3 \\ 5 & -1 & 1 \end{bmatrix}$$

The system of linear eq. can now be written

$$\boxed{Ax = b} \qquad \text{where} \quad b = \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}.$$

Suppose I do Gaussian elimination like Friday and arrive at the row eschelon ~~for~~ of A. which I'll call U. It turn out that we can factor A so that $\boxed{A = LU.}$

Thus $Ax = b$ can be written $L(Ux) = b$

substitute $y = Ux$ plug in to get $Ly = b$

rewrote

System $\left\{\begin{array}{l} Ly = b \\ Ux = y \end{array}\right.$

This makes sense, even if you don't know what matrix-matrix mult. is...

only involves matrix-vector operations...

Let's back up and do this carefully...

$f(x) = Ax$ is a linear function

Give any other matrices L and U, I could define more lin. functions.

$g(x) = Ux$

$h(y) = Ly$

need to check $g(x)$ satisfies

$g(a+b) = g(a) + g(b)$

$g(\alpha a) = \alpha g(a)$

Yes. Because matrix-vector mult is how to represent lin fn's.

Try composing the functions

$$(h \circ g)(x) = h(g(x)) = L\, g(x) = L(Ux)$$

$\boxed{\text{define matrix–matrix mult so that } LU \text{ is the matrix corresponding to the composition of functions } h \circ g.}$

for this definition to make sense... it's important for $h \circ g$ to be a linear function...

Do I get another lin function back when composing two lin. function? YES

Example:

$$\left.\begin{aligned} g(x) &= 3x \\ h(y) &= 4y \end{aligned}\right\} \text{ simplest lin functions.}$$

$$(h \circ g)(x) = h(g(x)) = 4\, g(x) = 4(3x)$$

$$= 12x$$

thats a linear function!