Correspondance between linear functions & Matrices...

linear function

① $f(x+y) = f(x) + f(y)$

② $f(\alpha x) = \alpha f(x)$

Suppose $\boxed{f : \mathbb{R}^n \to \mathbb{R}^m}$ and f is linear...

$x \in \mathbb{R}^n$ then

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_2} + \cdots + x_n \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{e_n}$$

Since f is linear

$$f(x) = x_1 f\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + x_2 f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + \cdots + x_n f\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right)$$

$$f(x) = \underbrace{x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n)}_{\text{actually a Matrix-vector multiplication}}$$

Note that $f(e_1)$ is a vector... $\boxed{f: \mathbb{R}^n \to \mathbb{R}^m}$

actualy $f(e_1)$ is an m-vector

Notation $f(e_1) \in \mathbb{R}^m$

Let

$$A = \begin{array}{cc} & \text{domain} \\ & n \text{ cols} \\ \begin{array}{c}\text{range} \\ m \text{ rows}\end{array} & \left[\begin{array}{c|c|c|c} f(e_1) & f(e_2) & \cdots & f(e_n) \end{array}\right] \end{array} \in \mathbb{R}^{m \times n} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$f(x) = Ax = \left[\begin{array}{c|c|c|c} f(e_1) & f(e_2) & \cdots & f(e_n) \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= f(e_1)\,x_1 + f(e_2)\,x_2 + \cdots + f(e_n)\,x_n$$

<u>Example</u>: $f: \mathbb{R}^3 \to \mathbb{R}^2$

Define $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 - x_3 \\ -1x_1 + 4x_2 + 2x_3 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Just knowing $f(e_1)$, $f(e_2)$ & $f(e_3)$ allows me to find A.

$$f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\cdot 1 + 3\cdot 0 - 0\\ -1\cdot 1 + 4\cdot 0 + 2\cdot 0\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix}, \quad f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\2\end{bmatrix}$$

$$A = \begin{bmatrix} f(e_1) & f(e_2) & f(e_3) \end{bmatrix} = \begin{array}{cc} & \text{3 cols} \\ \text{2 rows} & \begin{bmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \end{bmatrix} \end{array} \in \mathbb{R}^{2\times 3}$$

Then

$$f(x) = Ax$$

$$f\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_1 + 3x_2 - x_3\\ -1x_1 + 4x_2 + 2x_3\end{bmatrix}$$

Moving on to chapter 2 ...

2.1 Matrix-Matrix mult

2.2 Inverses ...

Let A be a matrix

Then $f(x) = Ax$ is a linear function

Want the inverse function say g such that

$$(f\circ g)(x) = x \quad \text{or} \quad (g\circ f)(x) = x$$

General talk about inverse functions

How about $f(x) = x^2$ and $g(x) = \sqrt{x}$

These are not linear function, but only here to give an example people already know

Domain of g is $[0, \infty)$

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (\sqrt{x})^2 = x$$

$$(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{x^2} = |x|$$