EXAM 1

Sept 29 (Friday) In class... attendance required...

$$A = \begin{array}{c|c} & \text{domain} \\ & n \text{ cols} \\ \begin{matrix}\text{range} \\ m \text{ rows}\end{matrix} & \begin{bmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \end{bmatrix} \end{array} \in \mathbb{R}^{m \times n} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Find the inverse function: Section 2.2

$$A^{-1} = \begin{bmatrix} f^{-1}(e_1) & f^{-1}(e_2) & \cdots & f^{-1}(e_n) \end{bmatrix}$$

So I need to find $f^{-1}(e_1), f^{-1}(e_2), \ldots, f^{-1}(e_n)$

By definition since $f(x) = Ax$ and $f^{-1}(y)$ is the value of x such that $f(x) = y$.

Need to solve $f(x) = e_1$, $f(x) = e_2$, ..., $f(x) = e_n$

$\boxed{Ax = e_1}$, $Ax = e_2$, ..., $Ax = e_n$

how to solve one of them

Augmented matrix

$[A | e_1]$

Augmented matrix

$[A | e_2]$

Augmented Matrix

$[A | e_n]$

All the elimination steps for reducing the augmented matrix are based on the matrix A, which is the same for each one.

Since the elimination steps are the same for each augmented matrix, put them all together

$$\left[\begin{array}{c|c|c|c|c} A & e_1 & e_2 & \cdots & e_n \end{array}\right]$$

so many different vectors on the right means that at least one of them is non-zero in each row... even after all the elimination steps...

Therefore A needs a pivot in each row so that $f(x) = e_k$ has a solution for each k.

- If A is square, the fact that there is a pivot in each row means there are no free variables.. This means there is a unique solution to $f(x) = e_k$ for each k.

That's what we need for an inverse!

Note if A is not square the either there are not enough pivots, or there are some free variables. That means there isn't an inverse.

$$\left[\begin{array}{c|c|c|c|c} A & e_1 & e_2 & \cdots & e_n \end{array}\right]$$

reduced row eschelon form

$$\left[\begin{array}{cccc|c|c|c|c} 1 & 0 & \cdots & 0 & & & & \\ 0 & 1 & \ddots & \vdots & f^{-1}(e_1) & f^{-1}(e_2) & \cdots & f^{-1}(e_n) \\ \vdots & & \ddots & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \end{array}\right]$$

Example: Find the inverse of

31. $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|c:c:c} 1 & 0 & -2 & 1 & \boxed{0 \quad 0} \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$r_2 \leftarrow r_2 + 3r_1$

$r_3 \leftarrow r_3 - 2r_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \qquad r_3 \leftarrow r_3 + 3r_2$$

lucky　　eliminate these

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & \boxed{2} & 7 & 3 & 1 \end{array}\right] \qquad \begin{array}{l} r_1 \leftarrow r_1 + r_3 \\ r_2 \leftarrow r_2 + r_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \qquad r_3 \leftarrow \frac{1}{2} r_3$$

$$\left[\begin{array}{ccc|c:c:c} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

$f^{-1}(e_1)$　　$f^{-1}(e_2)$　　$f^{-1}(e_3)$

Therefore

$$A^{-1} = \left[\begin{array}{c|c|c} f^{-1}(e_1) & f^{-1}(e_2) & f^{-1}(e_3) \end{array}\right] = \left[\begin{array}{c:c:c} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{array}\right]$$

When the matrix A is square but not invertible, the algorithm demonstrated above failes with an inconsistent system. Please look at the lecture notes for the 11am class for such an example.

There are also examples in the textbook of square matrices which don't have inverses...