Notation for column vectors written sideways

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 17 \end{bmatrix} \in \mathbb{R}^4$$

also

$$x = (x_1, x_2, \cdots, x_n) \approx (1, 2, 3, 17) \in \mathbb{R}^4$$

Note: $(1, 2, 3, 17) \approx \begin{bmatrix} 1 \\ 2 \\ 3 \\ 17 \end{bmatrix}$

However

$$\begin{bmatrix} 1 & 2 & 3 & 17 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \\ 17 \end{bmatrix}$$

related to the idea of a transpose...

Transpose of a matrix:

Example

notation...

$$\begin{bmatrix} 1 & 2 & 3 & 17 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 17 \end{bmatrix}$$

really a vector...

Idea: Switch the rows with columns...

Example :

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & -1 \\ 1 & 0 & -3 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 7 & 0 \\ 3 & -1 & -3 \end{bmatrix}$$

What does transpose mean in terms of matrix algebra?

What was the matrix in the first place?

Linear function $f: \mathbb{R}^n \to \mathbb{R}^m$

domain range

$A \in \mathbb{R}^{m \times n}$ $f(x) = Ax$

rows cols

Define a new function

$g(z) = f(x) \cdot z$ dot product

both vectors in the dot prod. must have the same length...

Thus $z \in \mathbb{R}^m$

$g: \mathbb{R}^m \to \mathbb{R}$ linear function...

Whats the matrix for g?

$$B = \left[g(e_1) \middle| g(e_2) \middle| \cdots \middle| g(e_m) \right]$$

Compute

$g(e_1) = g\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = f(x) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ = first component of $f(x)$...

$g(e_2) =$ second component of $f(x)$ ...

$\vdots$

so on

Thus

$$B = \begin{bmatrix} f(x) \text{ as a row} \end{bmatrix} = \begin{bmatrix} f(x)^T \end{bmatrix}$$

and

$$g(z) = f(x) \cdot z = f(x)^T z$$

as matrix multiplication

That was the easy part...

<u>Question</u>: is there a function $h(z)$ such that

$$g(z) \overset{?}{=} x \cdot h(z)$$

same $g(z)$ as before

Is there an $h(z)$ such that

$$f(x) \cdot z = \underbrace{x}_{\mathbb{R}^n} \cdot h(z) \text{ for all } x \text{ and } z.$$

Yes... but what is $h(z)$?

$$h : \mathbb{R}^{\boxed{m}} \to \mathbb{R}^{\boxed{n}}$$

from before

$f : \mathbb{R}^n \to \mathbb{R}^m$

domain range

$f(x) = Ax$

$x \in \mathbb{R}^n$

$g(z) = f(x) \cdot z$

o-p

Thus $z \in \mathbb{R}^m$

$g : \mathbb{R}^m \to \mathbb{R}$ linear function.

It's not a coincidence that the transpose switches the rows with the columns...

$$h(z) = A^T z$$

another name for A^T is the adjoint.

In summary

$$f(x) \cdot z = x \cdot h(z)$$

is the same as

$$Ax \cdot z = x \cdot A^T z$$

Catch up: Elementary row operation

1. Elimination step
2. row swap
3. rescaling