Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

$A =$ product of row operations

$A = E_1 E_2 E_3 \cdots E_{n-1} E_n I$

Then $AB = E_1 E_2 E_3 \cdots E_{n-1} E_n B$

$\det AB = \det E_1 E_2 E_3 \cdots E_{n-1} E_n B$

Theorem 3

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

$\det EI = \det E \det I$ ← since $\det I = 1$.

$\det EB =$ by theorem 3 $= \det E \det B$

Therefore ...

$\det AB = \det E_1 E_2 E_3 \cdots E_{n-1} E_n B$

$= \det E_1 \det E_2 E_3 \cdots E_{n-1} E_n B$

$= \det E_1 \det E_2 \det E_3 \cdots E_{n-1} E_n B$

$$= \left(\det E_1 \det E_2 \det E_3 \cdots \underbrace{\det E_{n-1} \det E_n}\right) \det B$$

$$= \left(\det E_1 \det E_2 \det E_3 \cdots \det E_{n-1} E_n\right) \det B$$

$$= \left(\det E_1 E_2 E_3 \cdots E_{n-1} E_n\right) \det B$$

$$= \det A \det B.$$

$\det A$ is a linear function of any one of its rows or columns

$T(c\mathbf{x}) = cT(\mathbf{x})$ for all scalars c and all $\mathbf{x}$ in $\mathbb{R}^n$

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$

linear func. in x.

$$T(x) = \det \overbrace{\begin{bmatrix} 1 & x_1 & 7 \\ 2 & x_2 & 8 \\ 3 & x_3 & 9 \end{bmatrix}}^{A} =$$

expand along this column using the definition

$$= \underbrace{(-1)^{1+2}}_{-1} x_1 \det A_{12} + x_2 \det A_{22} - x_3 \det A_{32}$$

$$= \underbrace{\begin{bmatrix} -\det A_{12} & \det A_{22} & -\det A_{32} \end{bmatrix}}_{\substack{\text{matrix} \\ 1\times 3}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\substack{\text{vector} \\ 3}}$$

multiplication is always a linear function.

8. $\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix} = \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$

$r_3 \leftarrow r_3 - 2r_1$

$r_4 \leftarrow r_4 + 3r_1$

Use Gaussian elimination

$$= \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{bmatrix}$$

$r_3 \leftarrow r_3 - r_2$

$r_4 \leftarrow r_4 + r_2$

$$= \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -15 \end{bmatrix}$$

$r_3 \leftrightarrow r_4$

$$= -\det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 10 \end{bmatrix} = -1 \cdot 1 \cdot 1 \cdot 10 = -10$$

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

Gaussian elimination to find each determinant..

How many determinants? $n+1$.

- from a practical point of view solving $Ax=b$ using Gaussian elimination just once is better...

- Theoretically its useful to have an equation for x...

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I_1(x) = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2 & 1 & 0 & 0 \\ x_3 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 1 \end{bmatrix} \qquad I_2(x) = \begin{bmatrix} 1 & x_1 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & x_3 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{bmatrix}$$

column swap

$c_1 \leftrightarrow c_2$

$$\det I_1(x) = x_1 \qquad \det I_2(x) = -\det \begin{bmatrix} x_1 & 1 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 1 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$

$$= \det \begin{bmatrix} x_2 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 \\ x_3 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 1 \end{bmatrix} = x_2$$

Similarly

$$\det I_3(x) = x_3 \quad \text{and} \quad \det I_4(x) = x_4$$

PROOF Denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and the columns of the $n \times n$ identity matrix I by $\mathbf{e}_1, \ldots, \mathbf{e}_n$. If $A\mathbf{x} = \mathbf{b}$, the definition of matrix multiplication shows that

$$\begin{aligned} A \cdot I_i(\mathbf{x}) &= A\begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{x} & \cdots & A\mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix} = A_i(\mathbf{b}) \end{aligned}$$

By the multiplicative property of determinants,

$$(\det A)(\det I_i(\mathbf{x})) = \det A_i(\mathbf{b})$$

divide through by det(A)

The second determinant on the left is simply x_i. (Make a cofactor expansion along the ith row.) Hence $(\det A) \cdot x_i = \det A_i(\mathbf{b})$. This proves (1) because A is invertible and $\det A \neq 0$. ∎

$$x_i = \det I_i(x) = \frac{\det A_i(b)}{\det A}$$

$$A\begin{bmatrix} \mathbf{e}_1 & e_2 \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{e}_1 & Ae_2 \cdots & \underbrace{A\mathbf{x}}_{b} & \cdots & A\mathbf{e}_n \end{bmatrix}$$

$$\boxed{A\, I_i(x) = A_i(b)}$$

identity with the ith column replaced by x

A with the ith column replaced by b.

$$\boxed{\det A \det I_i(x) = \det A I_i(x)} = \det A_i(b)$$

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

Once you have a way to solve $Ax=b$, then you have a related way to find A^{-1}.

How, just solve $Ax = e_i$ for $i=1,\ldots,n$.

With guassian elimination

$[A \mid b]$ then instead $[A \mid e_1 \mid e_2 \mid \ldots \mid e_n] = [A \mid I]$

with Cramer's Rule solve

$Ax = e_j$ using $x_i = \dfrac{\det A_i(e_j)}{\det A}$

s rule,

$$\{(i, j)\text{-entry of } A^{-1}\} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}$$

appear reversed from how the entries in a matrix are usually indexed...

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$