$$S = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$$

Then

$$\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}^{\alpha} = S D^{\alpha} S^{-1} = S \begin{bmatrix} 1^{\alpha} & 0 \\ 0 & 5^{\alpha} \end{bmatrix} S^{-1}$$

Check for finding the square root of a matrix

$$A^{1/2} = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}^{1/2} = S \cdot \begin{bmatrix} 1^{1/2} & 0 \\ 0 & 5^{1/2} \end{bmatrix} S^{-1}$$

Check that $A^{1/2} A^{1/2} = A$.

$$A^{1/2} A^{1/2} = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}^{1/2} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}^{1/2}$$

$$= S \cdot \begin{bmatrix} 1^{1/2} & 0 \\ 0 & 5^{1/2} \end{bmatrix} \cancel{S^{-1}} \cancel{S} \cdot \begin{bmatrix} 1^{1/2} & 0 \\ 0 & 5^{1/2} \end{bmatrix} S^{-1}$$

$$= S \cdot \begin{bmatrix} 1^{1/2} & 0 \\ 0 & 5^{1/2} \end{bmatrix} \cdot \begin{bmatrix} 1^{1/2} & 0 \\ 0 & 5^{1/2} \end{bmatrix} S^{-1}$$

$$= S \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} S^{-1} = A$$

What is a square root?

Solution to $x^2 - 2 = 0$ is $x = \pm\sqrt{2}$

plus or minus...

Check by plugging it in

$$(\sqrt{2})^2 - 2 = 0$$

$$(\sqrt{2})(\sqrt{2}) - 2 = 0$$

What is the solution to $\boxed{x^2 + 1 = 0}$? $x = \pm i$

$x = i$ then $i^2 = i \cdot i = -1$ by definition...

Quadratic equations have in general two solutions.
if $x = i$ is one of them, what's the other? $-i$

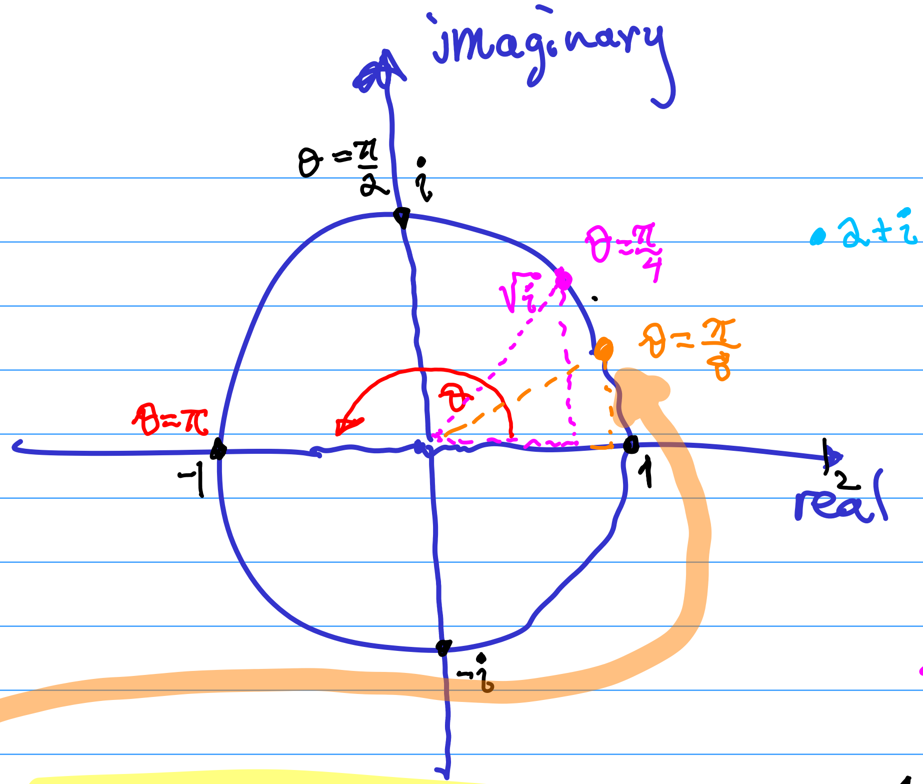
What's the complex conjugate of a complex number?

The operation that maps i to $-i$

- Note that one root of $x^2 + 1 = 0$ is as good as the other as long as you are consistent...

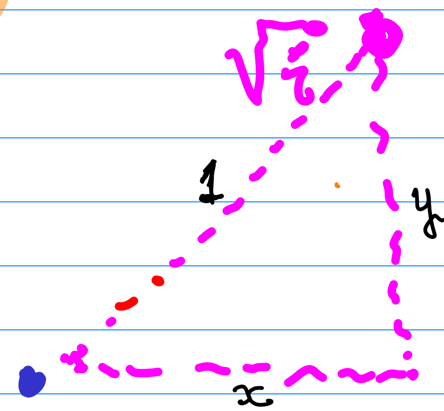
Note we often write $i = \sqrt{-1}$, Now what is $\sqrt{i}$?

Complex plane



$$\boxed{\sqrt{i} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}}$$

$$\sqrt{\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}} = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$$



note $x = y$.

$x^2 + y^2 = 1^2$

$2x^2 = 1$

$x = \frac{1}{\sqrt{2}}$

Given $A \in \mathbb{R}^{n \times n}$ we try to find eigenvalues λ and eigenvectors x such that

$$Ax = \lambda x$$

other side

How?

$$(A-\lambda I)x = 0$$

find vectors x which are not zero.

- That means $\text{Nul}(A-\lambda I)$ contains non-zero vectors
- That means $A-\lambda I$ has free variables...
- That means $\det(A-\lambda I)=0$.

solve this polynomial equation for λ.

When solving $\det(A-\lambda I)=0$ the values of λ might turn out to be complex...

There is a condition of $A \in \mathbb{R}^{n\times n}$ that guarantees the λ's are real:

$$A^T = A$$

We say A is symmetric...

one eigenvector

the other eigenvector

$$S = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$$

let's invert S...

$$\left[\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right] \qquad r_2 \leftarrow r_2 + \frac{3}{2} r_1$$

$$\left[\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ 0 & -2 & 3/2 & 1 \end{array}\right] \qquad r_1 \leftarrow r_1 - r_2$$

$$\left[\begin{array}{cc|cc} 2 & 0 & -1/2 & -1 \\ 0 & -2 & 3/2 & 1 \end{array}\right] \qquad \begin{array}{l} r_1 \leftarrow \frac{1}{2} r_1 \\ r_2 \leftarrow \frac{-1}{2} r_2 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1/4 & -1/2 \\ 0 & 1 & -3/4 & -1/2 \end{array}\right] \qquad \text{so} \quad S^{-1} = \begin{bmatrix} -1/4 & -1/2 \\ -3/4 & -1/2 \end{bmatrix}$$

- Note, I didn't check before that S^{-1} exists, but if there are two eigenvectors corresponding to two different eigenvalues, then they must be linearly independent.

→ Try to read this proof from section 5.1

M 2 If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

PROOF Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent. Since $\mathbf{v}_1$ is nonzero, Theorem 7 in Section 1.7 says that one of the vectors in the set is a linear combination of the preceding vectors. Let p be the least index such that $\mathbf{v}_{p+1}$ is a linear combination of the preceding (linearly independent) vectors. Then there exist scalars $c_1, \ldots, c_p$ such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{v}_{p+1} \tag{5}$$

Multiplying both sides of (5) by A and using the fact that $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ for each k, we obtain

$$\begin{aligned} c_1A\mathbf{v}_1 + \cdots + c_pA\mathbf{v}_p &= A\mathbf{v}_{p+1} \\ c_1\lambda_1\mathbf{v}_1 + \cdots + c_p\lambda_p\mathbf{v}_p &= \lambda_{p+1}\mathbf{v}_{p+1} \end{aligned} \tag{6}$$

Multiplying both sides of (5) by λ_{p+1} and subtracting the result from (6), we have

$$c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p = \mathbf{0} \tag{7}$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, the weights in (7) are all zero. But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct. Hence $c_i = 0$ for $i = 1, \ldots, p$. But then (5) says that $\mathbf{v}_{p+1} = \mathbf{0}$, which is impossible. Hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ cannot be linearly dependent and therefore must be linearly independent. ■

Jump ahead to 7.1. and assume

$A^T = A$ that is that A is symmetric...

what to show the eigenvalues of A are real...

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

and $A = A^T$ means $b = c$.

so $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

$$\chi(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix}$$

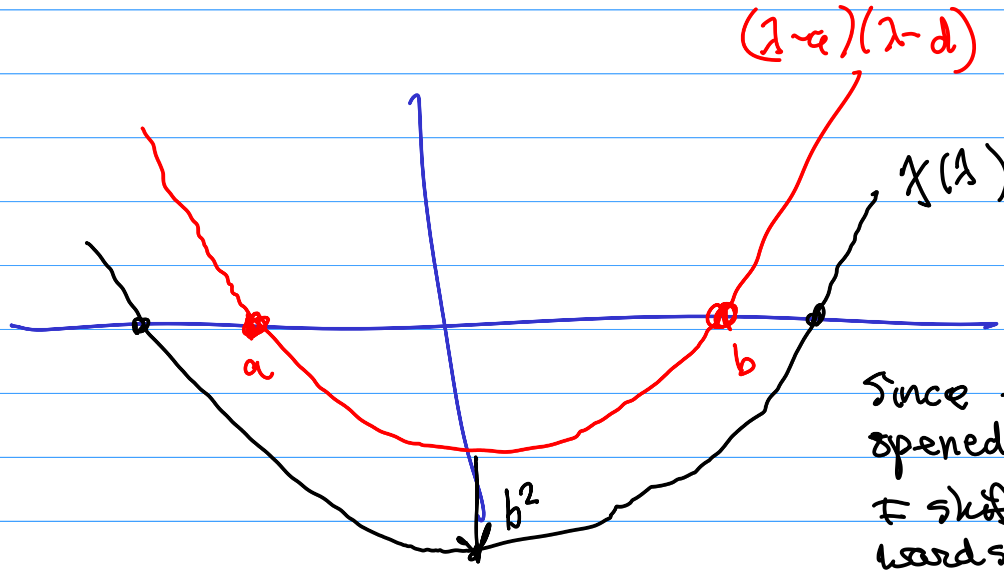
$$= (a-\lambda)(d-\lambda) - b^2 = 0$$

Solve for the λ's ....

Graph $\chi(\lambda) = \boxed{(\lambda - a)(\lambda - d)} - b^2$

this part parabola passing through a and d which opens upward.

shifted downward by b^2



. . .

Since the parabola opened upwards and I shifted it downwards, then it still intersects the x-axis ...

$\downarrow$

Thus $\chi(\lambda)=0$ has real solutions for λ.