The **length** (or **norm**) of **v** is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$v \in \mathbb{R}^n$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$$

the entries of v are real so their squares are positive...

$x \in \mathbb{C}^n$

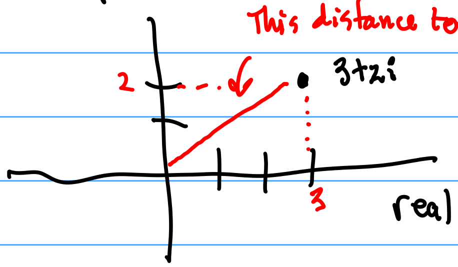
$$\|x\| = \sqrt{\bar{x}\cdot x} = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

since the entries of x are complex we add the absolute values (modulus) so that these terms are positive...

$z = 3+2i$ $\qquad \bar{z} = 3-2i$

$$\boxed{\bar{z}\cdot z} = (3-2i)(3+2i) = 9 \cancel{+6i} \cancel{-6i} - 4i^2 = 9+4 = 13 = |z|^2$$

This distance to the origin is $\sqrt{3^2+2^2} = \sqrt{13}$



• No more complex numbers for some time...

6.4 THE GRAM–SCHMIDT PROCESS

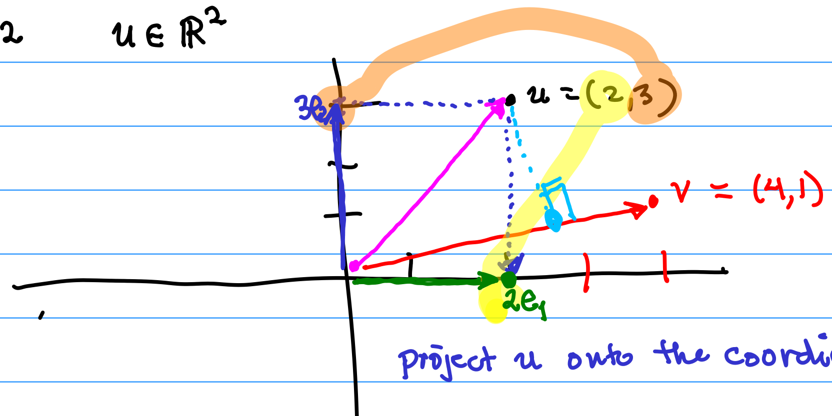
Main Idea, but first

Interested in projections:

$n=2$ $\qquad u \in \mathbb{R}^2$

$\|u\| = \sqrt{2^2+3^2} = \sqrt{13}$

$\|v\| = \sqrt{4^2+1^2} = \sqrt{17}$



$e_1 = (1,0)$

$e_2 = (0,1)$

$\|e_1\| = \sqrt{1^2+0^2} = 1$

$\|e_2\| = \sqrt{0^2+1^2} = 1$

project u onto the coordinate axes...

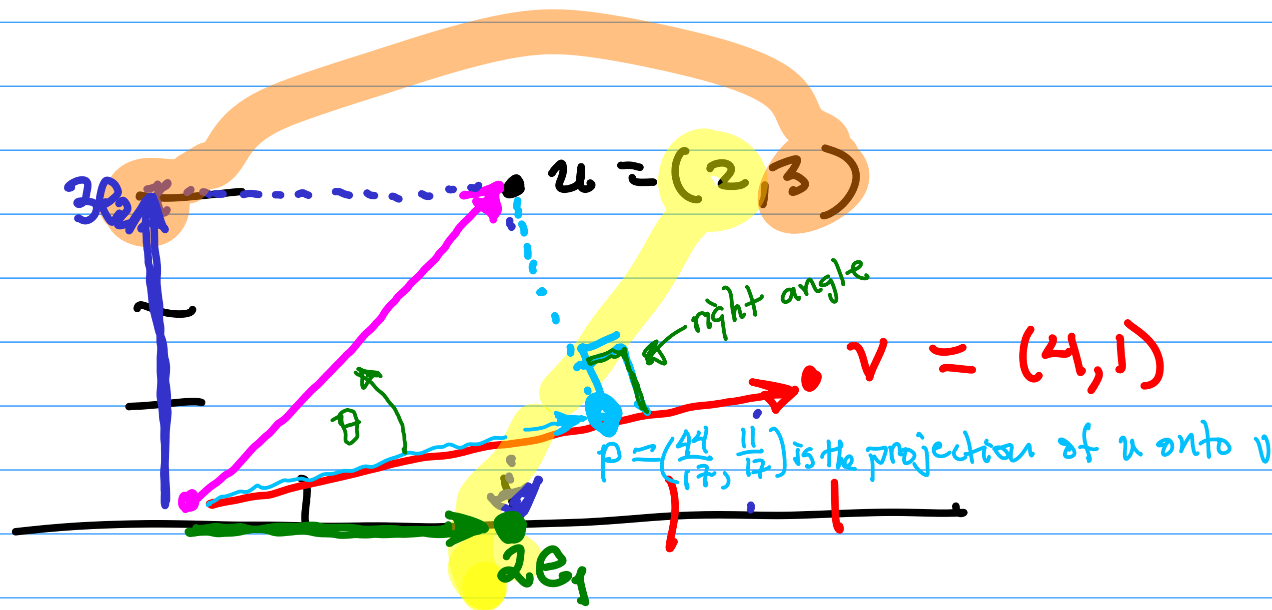
projection onto the e_1 axis is given by $u \cdot e_1 = (2,3)\cdot(1,0) = 2$ ← The distance along the axis

The actual projection is $2e_1$

e_2 axis is given by $u \cdot e_2 = (2,3)\cdot(0,1) = 3$ ← distance along the vertical axis

The actual projection is $3e_2$

Now consider projection of u onto the vector v.



How to find this projection? use dot product...

$$\hat{v} = \frac{v}{\|v\|} = \frac{(4,1)}{\sqrt{17}}$$ unit vector in the v direction...

Check it's a unit vector... ☑

$$\|\hat{v}\| = \sqrt{\hat{v}\cdot\hat{v}} = \sqrt{\frac{v}{\|v\|}\cdot\frac{v}{\|v\|}} = \sqrt{\frac{v\cdot v}{\|v\|^2}} = \frac{\sqrt{v\cdot v}}{\|v\|} = \frac{\|v\|}{\|v\|} = 1$$

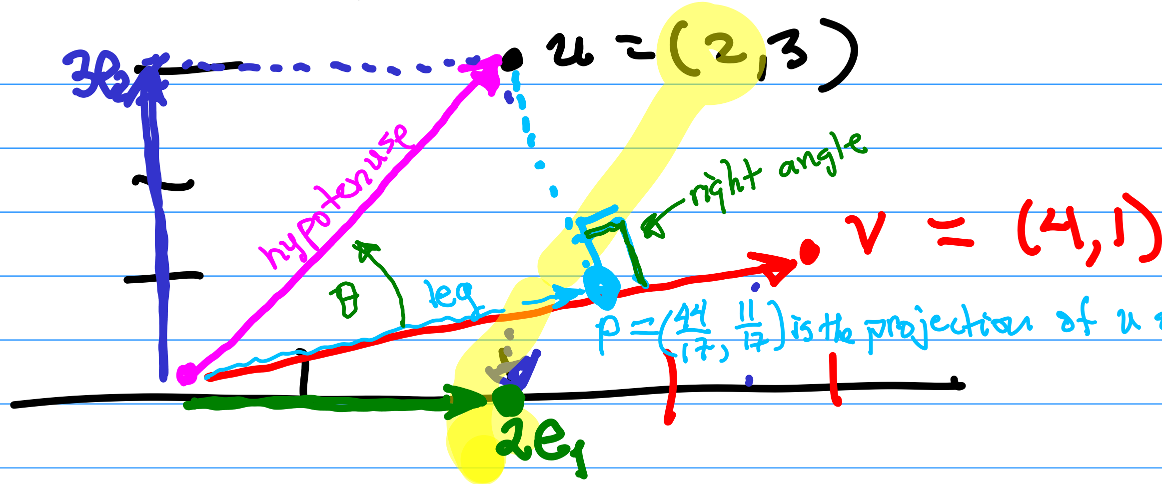
Then the distance of the projection from the origin is in the v direction

$$\|p\| = u\cdot\hat{v} = (2,3)\cdot\frac{(4,1)}{\sqrt{17}} = \frac{8+3}{\sqrt{17}} = \frac{11}{\sqrt{17}}$$

Thus

$$p = \|p\|\,\hat{v} = \frac{11}{\sqrt{17}}\cdot\frac{(4,1)}{\sqrt{17}} = \frac{11}{17}(4,1) = \left(\frac{44}{17}, \frac{11}{17}\right)$$

Alternative approach using trigonometry...



$$\cos\theta = \frac{\text{leg}}{\text{hypotenuse}} = \frac{\|p\|}{\|u\|}$$

Thus

$$\|p\| = \|u\|\cos\theta \quad \text{also} \quad \|p\| = u \cdot \hat{v} = u \cdot \frac{v}{\|v\|}$$

Therefore

$$\|u\|\cos\theta = \frac{u \cdot v}{\|v\|} \quad \text{or} \quad u \cdot v = \|u\|\|v\|\cos\theta .$$

Maybe better to think about this as a way of finding θ.

$$\cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|}$$

The projection formula for p works in any dimension as is

$$p = \|p\|\hat{v} = (u \cdot \hat{v})\hat{v} = \left(u \cdot \frac{v}{\|v\|}\right)\frac{v}{\|v\|} = \boxed{\frac{(u \cdot v)\, v}{\|v\|^2}}$$

Application of this projection formula is the algorithm called Gram-Schmidt orthogonalization...

Idea... Given a matrix $A \in \mathbb{R}^{m \times n}$ with linearly independent columns, factor the matrix as

$$A = QR$$

triangular

satisfy the orthogonality property $Q^TQ = I$

• If Q were a square matrix (which happens when A is square) then $Q^TQ = I$, implies $Q^{-1} = Q^T$

How to find Q

6.4 THE GRAM–SCHMIDT PROCESS

Main Idea, but first

$$A = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}$$

where

$$u_1 = x_1 \qquad V_1 = \frac{u_1}{\|u_1\|}$$

$$u_2 = x_2 - (V_1 \cdot x_2)V_1 \qquad V_2 = \frac{u_2}{\|u_2\|}$$

Claim that V_1 and V_2 are perpendicular...

$$V_1 \cdot V_2 = V_1 \cdot \frac{u_2}{\|u_2\|} = \frac{1}{\|u_2\|} V_1 \cdot u_2$$

$$= \frac{1}{\|u_2\|} V_1 \cdot \left(x_2 - (V_1 \cdot x_2) V_1 \right)$$

$$= \frac{1}{\|u_2\|} \left(V_1 \cdot x_2 - (V_1 \cdot x_2)(V_1 \cdot V_1) \right) = 0$$

Want the third column to be perpendicular to the other two ... after some column operations

$$u_3 = x_3 - (v_1 \cdot x_3)v_1 - (v_2 \cdot x_3)v_2 \qquad v_3 = \frac{u_3}{\|u_3\|}$$

all these thing are column operations...

$\vdots$

$$u_n = x_n - (v_1 \cdot x_n)v_1 - \cdots - (v_{n-1} \cdot x_n)v_{n-1} \qquad v_n = \frac{u_n}{\|u_n\|}$$

We'll figure out R next time... (put them in R)

What about Q? $n=2$

$$A = \begin{bmatrix} x_1 \mid x_2 \end{bmatrix} \quad \cdots \quad Q = \begin{bmatrix} v_1 \mid v_2 \end{bmatrix}$$

$v_1 \cdot v_1 = 1 \qquad v_1 \cdot v_2 = 0$

$v_2 \cdot v_1 = 0 \qquad v_2 \cdot v_2 = 1$

$$Q^T Q = \begin{bmatrix} v_1^T \\ \hline v_2^T \end{bmatrix} \begin{bmatrix} v_1 \mid v_2 \end{bmatrix} = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$