The Gram-Schmidt process...

$u_1 = x_1$ $\qquad$ $v_1 = \frac{u_1}{\|u_1\|}$

$\hookrightarrow x_1 = u_1 = \|u_1\| v_1$

remember this

$u_2 = x_2 - (v_1 \cdot x_2) v_1$ $\qquad$ $v_2 = \frac{u_2}{\|u_2\|}$

$x_2 = (v_1 \cdot x_2) v_1 + u_2 = \underline{(v_1 \cdot x_2)} v_1 + \underline{\|u_2\|} v_2$

$u_3 = x_3 - (v_1 \cdot x_3) v_1 - (v_2 \cdot x_3) v_2$ $\qquad$ $v_3 = \frac{u_3}{\|u_3\|}$

$x_3 = (v_1 \cdot x_3) v_1 + (v_2 \cdot x_3) v_2 + u_3 = \underline{(v_1 \cdot x_3)} v_1 + \underline{(v_2 \cdot x_3)} v_2 + \underline{\|u_3\|} v_3$

$u_n = x_n - (v_1 \cdot x_n) v_1 - \cdots - (v_{n-1} \cdot x_n) v_{n-1}$ $\qquad$ $v_n = \frac{u_n}{\|u_n\|}$

$x_n = \underline{(v_1 \cdot x_n)} v_1 + (v_2 \cdot x_n) v_2 + \cdots + (v_{n-1} \cdot x_n) v_{n-1} + \|u_n\| v_n$

$$A = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}, \quad Q = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}, \quad R = \begin{bmatrix} \|u_1\| & v_1 \cdot x_2 & v_1 \cdot x_3 & \cdots & v_1 \cdot x_n \\ 0 & \|u_2\| & v_2 \cdot x_3 & \cdots & v_2 \cdot x_n \\ 0 & 0 & \|u_3\| & & \vdots \\ 0 & 0 & 0 & \ddots & v_{n-1} \cdot x_n \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \|u_n\| \end{bmatrix}$$

row column

upper triangular

$A = QR$

EXAMPLE

6.4 #10

$$\begin{array}{c} A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \\ \quad x_1 \quad\; x_2 \quad\; x_3 \end{array}, \quad Q = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad R = \begin{bmatrix} 2\sqrt{3} & -6\sqrt{3} & \sqrt{3} \\ 0 & 2\sqrt{3} & 5\sqrt{3} \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$$

don't multiply it out...

$$u_1 = x_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \qquad v_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{1+9+1+1}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = x_2 - (v_1 \cdot x_2) v_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \right) \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{-6-24-2-4}{2\sqrt{3}} \right) \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{-18\sqrt{3}}{\sqrt{3}\sqrt{3}} \right) \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(-6\sqrt{3} \right) \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \qquad v_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{9+1+1+1}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$u_3 = x_3 - (v_1 \cdot x_3) v_1 - (v_2 \cdot x_3) v_2$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \left(\frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \right) \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \right) \frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \left(\frac{-6+9+6-3}{2\sqrt{3}} \right) \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{18+3+6+3}{2\sqrt{3}} \right) \frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - (\sqrt{3}) \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - (5\sqrt{3}) \frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$\frac{12}{2} + \frac{1}{2} - \frac{15}{2} = \frac{-2}{2} = -1$

$\frac{6}{2} - \frac{3}{2} - \frac{5}{2} = \frac{-2}{2} = -1$

$\frac{12}{2} - \frac{1}{2} - \frac{5}{2} = \frac{6}{2} = 3$

$-\frac{6}{2} - \frac{1}{2} + \frac{5}{2} = \frac{-2}{2} = -1$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$$V_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{1+1+9+1}} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Therefore

$$V_1 = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad V_2 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad V_3 = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

and

$$Q = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 & 3 & -1 \\ 3 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix} \qquad R = \begin{bmatrix} 2\sqrt{3} & -6\sqrt{3} & \sqrt{3} \\ 0 & 2\sqrt{3} & 5\sqrt{3} \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$$

Check if $A = QR$

$$\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 & 3 & -1 \\ 3 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & -6\sqrt{3} & \sqrt{3} \\ 0 & 2\sqrt{3} & 5\sqrt{3} \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$$

$$\frac{1}{2\sqrt{3}}\begin{bmatrix}3\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}\sqrt{3}\\5\sqrt{3}\\2\sqrt{3}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}3\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\5\\2\end{bmatrix}=\frac{3+5-2}{2}=\frac{6}{2}=3$$

Applications:

Solve $Ax=b$ $\qquad A=LU$

triangular matrices

$LUx=b$

solve the system $\begin{cases} Ly=b \\ Ux=y \end{cases}$

solve each system by substitution because L and U are triangular...

$Ax=b$ $\qquad A=QR$

triangular

orthogonality property.

$Q^TQ=I$

$QRx=b$

$Q^TQRx=Q^Tb$

$Rx=Q^Tb$ Matrix vector multiplication is even easier...

triangular so can solve for x using substitution

Original problem:

$$\begin{bmatrix}-1 & 6 & 6\\3 & -8 & 3\\1 & -2 & 6\\1 & -4 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

In the best case, there is a pivot in every column but since there are more rows than there is not a pivot in every rows...

Sometimes you get $0=$ non-zero and the system is inconsistent.

inconsistent for most b's...

$$Rx = Q^T b$$

$$Q^T = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 & 3 & 1 & 1 \\ 3 & 1 & 1 & -1 \\ -1 & -1 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2\sqrt{3} & -6\sqrt{3} & \sqrt{3} \\ 0 & 2\sqrt{3} & 5\sqrt{3} \\ 0 & 0 & 2\sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 & 3 & 1 & 1 \\ 3 & 1 & 1 & -1 \\ -1 & -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Note $Rx = Q^T b$ always has a solution no matter what b is...