$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

$$B = A^T A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

Find eigenvalues and eigenvectors of B. By the spectral theorem since $B^T = B$ then the eigenvectors can be chosen to be orthonormal and the eigenvalues are real and in this, even non-negative.

Traditionally we order the eigenvalues from biggest to smallest...

$\lambda_1 = 9, \quad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\qquad \lambda_2 = 4, \quad x_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\|x_1\| = \sqrt{5}$ $\qquad \|x_2\| = \sqrt{5}$

$$U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Trying to find the factorization of $A = V \Sigma U^T$

What's V?

$$z_1 = \frac{Ax_1}{\|Ax_1\|} = \frac{y_1}{\sqrt{\lambda_1}} = \frac{1}{\cancel{3}}\frac{1}{\sqrt{5}}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$z_2 = \frac{Ax_2}{\|Ax_2\|} = \frac{y_2}{\sqrt{\lambda_2}} = \frac{1}{2}\frac{1}{\sqrt{5}}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

Matrices in the singular value decomposition of A..

$$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Check:

diagonal

$$A = V\Sigma U^T = \left[\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\right]\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

orthogonal

$$= \begin{bmatrix} 3/\sqrt{5} & -4/\sqrt{5} \\ 6/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 10 & -5 \\ 10 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

Worked! since $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$

What could go wrong? A might not have linearly ind. columns, in which case B would have a zero eigenvalue...

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

$$B = A^T A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$$

$$\det(B - \lambda I) = \det\begin{bmatrix} 2-\lambda & -4 \\ -4 & 8-\lambda \end{bmatrix} = (2-\lambda)(8-\lambda) - 16$$

$$= \lambda^2 - 10\lambda + 16 - 16 = \lambda(\lambda - 10) = 0$$

eigenvalues $\lambda_1 = 10$, $\lambda_2 = 0$.

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$$

Find eigenvectors:

$\lambda = 10$

$$\mathrm{Nul}(B - \lambda I) = \mathrm{Nul}\begin{bmatrix} -8 & -4 \\ -4 & -2 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$2x_1 + x_2 = 0$$

$$x_1 = -\tfrac{1}{2}x_2$$

free vbl

$$x = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

eigenvector for $\lambda_1 = 10$

set $x_2 = 2$ for convenience...

$\lambda = 0$

$$\mathrm{Nul}(B - \lambda I) = \mathrm{Nul}\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$x_1 - 2x_2 = 0$

$x_1 = 2x_2$

$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ eigenvector for $\lambda_2 = 0$

set $x_2 = 1$ for convenience

$U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ $\qquad \Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$

$$z_1 = \frac{Ax_1}{\|Ax_1\|} = \frac{y_1}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{10}}\frac{1}{\sqrt{5}}\begin{bmatrix} -5 \\ 5 \end{bmatrix} = \frac{1}{5\sqrt{2}}\begin{bmatrix} -5 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$y_1 = Ax_1 = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

$$y_2 = Ax_2 = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

make up something that is ☑

① a unit vector

② perpendicular to all the other columns...

Check it

$$V\Sigma U^T = \left(\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

singular value was zero

didn't play a role in finding A because the corresponding

$$= \begin{bmatrix} -\sqrt{10}/\sqrt{2} & 0 \\ \sqrt{10}/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

same as this

$$\begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} -\sqrt{10}/\sqrt{2} & 0 \\ \sqrt{10}/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Reduced singular value decomposion.

$$V\Sigma U^T = \left(\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$\boxed{= \left(\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{10} \end{bmatrix} \right) \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{2} \end{bmatrix}}$$

$$= \begin{bmatrix} -\sqrt{10}/\sqrt{2} \\ \sqrt{10}/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Another example

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

nd a unit vector x at w

Hint work with A^T ...

may not be square ...

diagonal

$$A = V \Sigma U^T$$

$$A^T = (V \Sigma U^T)^T = U^{TT} \Sigma^T V^T = U^T \Sigma^T V^T$$

orthogonal matrix

orthogonal

- Let's work with $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \in \mathbb{R}^{3\times 2}$ and then take transposes in the end to get back to the original matrix A.

$$\underset{3\times 2}{A} = \underset{3\times 3}{V} \underset{3\times 2}{\Sigma} \underset{2\times 2}{U^T}$$

$$B = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

only has two eigenvectors

$\det(B - \lambda I)$ $\qquad \lambda_1 \qquad \lambda_2$

$\mathrm{Nul}(B - \lambda I)$ $\qquad x_1 \in \mathbb{R}^2 \qquad x_2 \in \mathbb{R}^2$

$$\underset{3\times 2\ \ 2}{y_1 = A x_1} \in \mathbb{R}^3$$

$$\underset{3\times 2}{V} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}$$

only two z_i's because only two x_i's

Make up so V is square ... and orthogonal

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix}$$