From last time

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

$$B = A^T A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

Use spectral theorem to find eigenvalues and eigenvectors of B

$\lambda_1 = 9, \quad x_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \lambda_2 = 4 \quad x_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Thus

$$U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

What is V?

recall $y_i = A x_i$

$$\boxed{z_i = \frac{y_i}{\|y_i\|} = \frac{y_i}{\sqrt{\lambda_i}}}$$

$$y_1 = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 3 \\ 6 \end{bmatrix} \qquad z_1 = \frac{1}{3}\frac{1}{\sqrt{5}}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -4 \\ 2 \end{bmatrix} \qquad z_2 = \frac{1}{2} \frac{1}{\sqrt{5}} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Thus

$$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

and $A = V \Sigma U^T$

Check this

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{5} \left(\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 3 & -4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 & -5 \\ 10 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

same

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

Next example:

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$$

A does not have linearly ind columns and so B is not invertible

$$B = A^TA = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 20 & -10 \\ -10 & 5 \end{bmatrix} = 5 \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\det(B - \lambda I) = \det\left(5\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} - \lambda I\right)$$

$\alpha = \frac{\lambda}{5}$

$\lambda = 5\alpha$

$$= 5^2 \det\left(\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} - \frac{\lambda}{5} I\right)$$

$$\det\left(\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} - \alpha I\right) = \det\begin{bmatrix} 4-\alpha & -2 \\ -2 & 1-\alpha \end{bmatrix}$$

$$= (4-\alpha)(1-\alpha) - 4 = \alpha^2 - 5\alpha + 4 - 4$$

$$= \alpha(\alpha - 5) = 0$$

$\alpha = 0$ or $\alpha = 5$

$\lambda = 0$ or $\lambda = 25$

Find the eigenvectors using α (for computational simplicity)

$\lambda = 0$

$\alpha = 0$

$$\text{Nul}\left(\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} - \alpha I\right) = \text{Nul}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} = \text{Nul}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$2x_1 - x_2 = 0$

$x_1 = \frac{1}{2}x_2$

$$x = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

set $x_2 = 2$

unit eigen vector is $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = 25$

$\alpha = 5$

$$\text{Nul}\left(\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} - \alpha I\right) = \text{Nul}\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

unit eigenvector is $\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Check that factoring 5 ~~out~~ didn't lead to a mistake...

$$\begin{bmatrix} 20 & -10 \\ -10 & 5 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -50 \\ 25 \end{bmatrix} = 25\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} \sqrt{25} & 0 \\ 0 & \sqrt{0} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

orthogonal matrix ... columns are orthonormal and the matrix is square...

$$y_1 = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -5 \\ 10 \end{bmatrix}, \quad z_1 = \frac{y_1}{\sqrt{\lambda_1}} = \frac{1}{5}\frac{1}{\sqrt{5}}\begin{bmatrix} -5 \\ 10 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad z_2 = ?$$

Thus,

$$V = \begin{bmatrix} z_1 & ? \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

put anything in the missing columns that completes V to make an orthogonal matrix

Could also put the negative of this vector and everything would still be fine...

Check it really works

$$A = V \Sigma U^T$$

$$\left(\begin{bmatrix} -1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} -5/\sqrt{5} & 0 \\ 10/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 & -5 \\ -20 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$$

One more thing, since all the zeros one can simplify...

Reduced singular value decomposition...

$$\left(\begin{bmatrix} -1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

reduced SVD

$$= \boxed{\left(\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix} \right) \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}}$$

$$= \begin{bmatrix} -5/\sqrt{5} \\ 10/\sqrt{5} \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 & -5 \\ -20 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$$

13. Find the SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$ [*Hint:* Work with A^T.]

14. ~~In Exercise 7, find a unit vector **x** at which A**x** has maximum~~

Let $\underset{3\times 2}{A^T} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$

$\underset{3\times 2}{A^T} = \underset{3\times 3}{V}\,\underset{3\times 2}{\Sigma}\,\underset{2\times 2}{U^T}$ ← diagonal matrix will not be square

$A = (V\Sigma U^T)^T$

$= U^{TT}\Sigma^T V^T$.

$A = U\Sigma^T V^T$

singular value decomposition of A by working with A^T...

$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 2}$

still diagonal when transposed but changes shape...

$\Sigma^T = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \end{bmatrix} \in \mathbb{R}^{2\times 3}$

$\underset{2\times 2}{C} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$

2 eigenvalues and 2 eigenvectors of C...

$\Sigma \in \mathbb{R}^{3\times 2}$ $\quad U \in \mathbb{R}^{2\times 2}$

Find the matrix ~~V~~ V:

$y_i = \underset{3\times 2}{A^T}\,\underset{2}{x_i} \in \mathbb{R}^3$

$z_i = \dfrac{y_i}{\|y_i\|} = \dfrac{y_i}{\sqrt{\lambda_i}}$ $\quad i = 1, 2$

$$\underset{3\times 3}{V} = \left[\begin{array}{c|c|c} z_1 & z_2 & ? \end{array}\right] \in \mathbb{R}^{3\times 3}$$

pick a vector perp to the other two so that V is an orthogonal matrix.