Correspondance between matrices & linear functions:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 5 & -7 \\ 0 & 7 & -1 \end{bmatrix} \Longleftrightarrow f(x_1, x_2, x_3)$$

$$A\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}5\\-16\\-9\end{bmatrix} = f(1,-1,2)$$

1. Viewing matrix vector mult as a bunch of dot products...

$$\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 5 & -7 \\ 0 & 7 & -1 \end{bmatrix} \begin{bmatrix}5\\-16\\-9\end{bmatrix}$$

Two different ways
to compute the
matrix-vector product

2. View as linear combinations of vectors...

$$A\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}1\\3\\0\end{bmatrix}\cdot 1 + \begin{bmatrix}-2\\5\\7\end{bmatrix}(-1) + \begin{bmatrix}1\\-7\\-1\end{bmatrix}(2)) = \begin{bmatrix}5\\-16\\-9\end{bmatrix}$$

EOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $\boxed{A\mathbf{x} = \mathbf{b}}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$. ← everything

d. A has a pivot position in every row. ← why is this the same?

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 5 & -7 \\ 0 & 7 & -1 \end{bmatrix} = \begin{bmatrix} a_1 \,\big|\, a_2 \,\big|\, a_3 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} -2 \\ 5 \\ 7 \end{bmatrix}$$

$$a_3 = \begin{bmatrix} 1 \\ -7 \\ -1 \end{bmatrix}$$

The span is just a notation for things we already know...

$$\text{Span}(a_1, a_2, a_3) = \{ a_1c_1 + a_2c_2 + a_3c_3 : c_1 \in \mathbb{R}, c_2 \in \mathbb{R}, c_3 \in \mathbb{R} \}$$

if any of these terms you know...then
the other mean exactly the same thing...

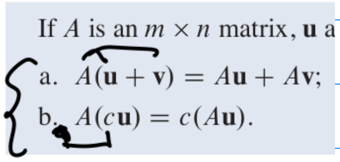
$$= \{ Ax : x \in \mathbb{R}^3 \} = \text{range}(f)$$

where $f(x) = Ax$.

If there is a pivot in every row then there is no row that's all zeros in the echelon form of A. which means there is no compatibility condition of the form $\boxed{0 = C}$ ← last time so any right hand side is okay.

THEOREM 5

margin notes made in the book during class

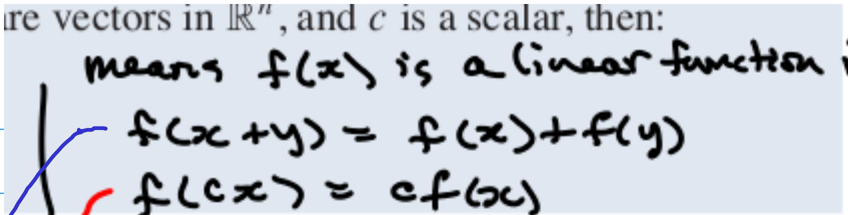


If A is an $m \times n$ matrix, $\mathbf{u}$ a

a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

definition of a linear function written in term of matrices

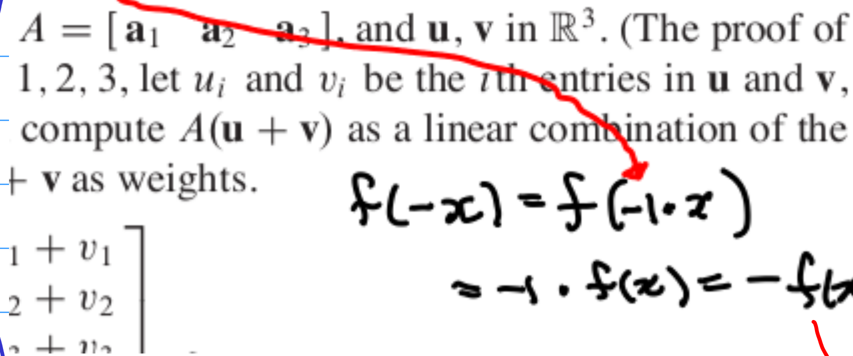


re vectors in $\mathbb{R}^n$, and c is a scalar, then:

means $f(x)$ is a linear function if

$f(x+y) = f(x) + f(y)$

$f(cx) = cf(x)$



$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$, and $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$. (The proof of 1, 2, 3, let u_i and v_i be the ith entries in $\mathbf{u}$ and $\mathbf{v}$, compute $A(\mathbf{u}+\mathbf{v})$ as a linear combination of the $+ \mathbf{v}$ as weights.

$_1 + v_1$

$_2 + v_2$

$$f(-x) = f(-1 \cdot x)$$
$$= -1 \cdot f(x) = -f(x)$$

in one dimension a linear function is a line that passes through the origin

$f(x) = 3x$

$f(2) + f(5) = 3 \cdot 2 + 3 \cdot 5$

$= 3(2+5) = f(2+5)$

$f(4 \cdot 2) = 3 \cdot 4 \cdot 2 = 4 \cdot 3 \cdot 2$

$= 4f(2)$

How to show f(0)=0 ?

$\boxed{y = -x}$

$$f(x+y) = f(x+(-x)) = f(0)$$
$$\|$$
$$f(x) + f(y) = f(x) + f(-x) = f(x) - f(x) = 0$$

what makes zero on the right side different than other right-hand sides is that the column stays zero no matter what row operations are done

Start of Section 1.5

Homogeneous equations $Ax = 0$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 3 & 5 & -7 & 0 \\ 0 & 7 & -1 & 0 \end{array}\right]$$

$r_2 \leftarrow r_2 - 3r_1$

row col

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 11 & -10 & 0 \\ 0 & 7 & -1 & 0 \end{array}\right]$$

$r_3 \leftarrow r_3 - \frac{7}{11} r_2$

row col

focus on the matrix A is enough...

still zeros in this column...

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 11 & -10 \\ 0 & 0 & \frac{59}{11} \end{bmatrix} = U$$

$$-1-\frac{7}{11}(-10)$$

$$= -1+\frac{70}{11} = \frac{70-11}{11}$$

$$= \frac{59}{11}$$

$$A = LU$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 5 & -7 \\ 0 & 7 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 7/11 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 11 & -10 \\ 0 & 0 & \frac{59}{11} \end{bmatrix}$$

$$Ax = 0$$

$$LUx = 0$$

System of systems

$$\begin{cases} Ly = 0 \\ Ux = y \end{cases}$$

$$Ux = 0$$

To solve $Ax=0$ I only need to solve $Ux=0$

$$Ly = 0$$

because of the ones always on the diagonal of L, the solution is unique

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 7/11 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} y_1 = 0 \\ 3y_1 + y_2 = 0 \\ 7/11\, y_2 + y_3 = 0 \end{cases}$$

$$y_2 = -3y_1 = -3 \cdot 0 = 0$$

$$y_3 = -\frac{7}{11} y_2 = -\frac{7}{11} \cdot 0 = 0$$

(note that y=0 is always a solution to Ly=0)

The matrix L always has 1's on the diagonal which means the solution to $Ly=0$ is unique.

So $y=0$ is the only solution to $Ly=0$.

Another way to see that the solution to Ax=0 is given by Ux=0 is directly from the augmented matrix...

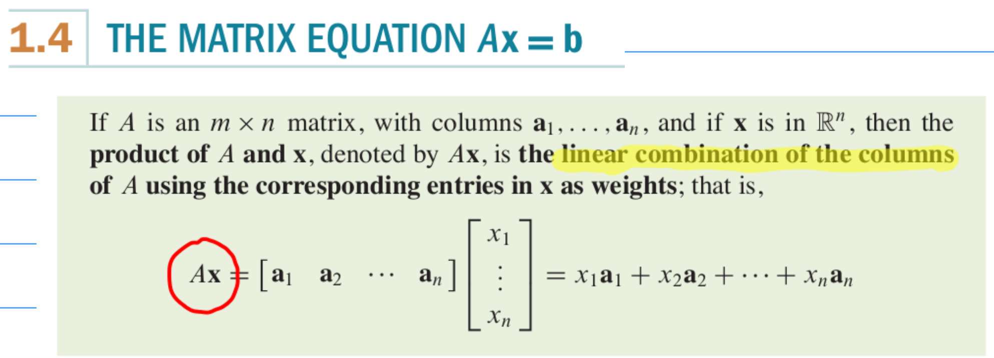
Back to the augmented matrix for $Ax=0$

after elimination we get this:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 11 & -10 & 0 \\ 0 & 0 & \frac{59}{11} & 0 \end{array}\right] = \left[\begin{array}{c|c} U & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right]$$

Solve is the same as solving $Ux=0$

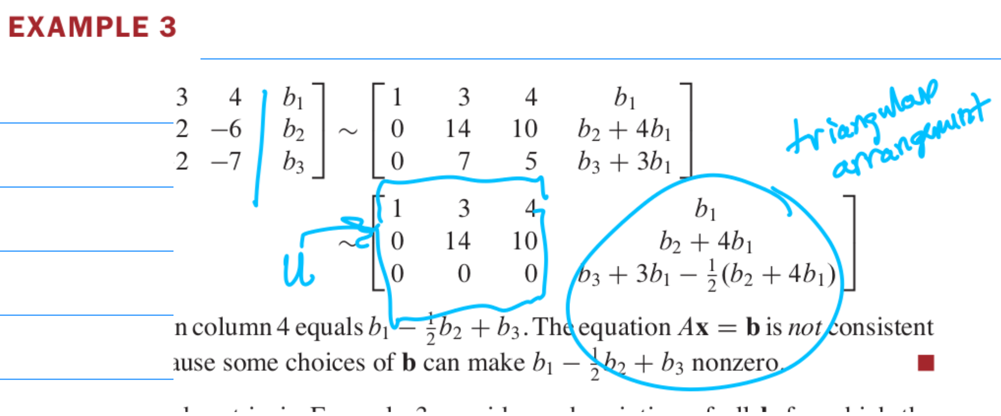
Some stuff highlighted from the book during the lecture...



1.4 THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, **is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$\boxed{A\mathbf{x}} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$



EXAMPLE 3

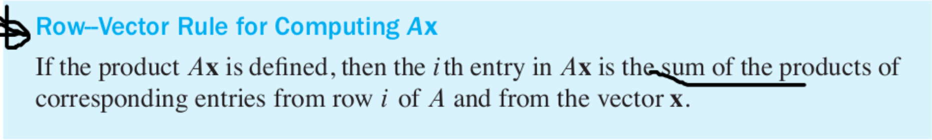
$$\left.\begin{array}{cc|c} 3 & 4 & b_1 \\ 2 & -6 & b_2 \\ 2 & -7 & b_3 \end{array}\right] \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\underset{U}{\sim} \begin{bmatrix} \boxed{\begin{matrix} 1 & 3 & 4 \\ 0 & 14 & 10 \\ 0 & 0 & 0 \end{matrix}} & \begin{matrix} b_1 \\ b_2 + 4b_1 \\ b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{matrix} \end{bmatrix}$$

triangular arrangement

n column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is *not* consistent ause some choices of $\mathbf{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero. ■

actually how we defined Ax in the lecture and obtained linear combinations as a consequence...



Row–Vector Rule for Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.