in the pivot position, but no good because it's zero..

swap rows ...

$$A = \begin{bmatrix} \boxed{0} & 1 & 2 & 0 & 3 & 10 \\ 1 & 2 & 8 & -1 & -2 & 5 \\ -2 & -3 & -14 & 3 & 8 & -2 \\ -3 & -4 & -20 & 4 & 15 & 9 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$

Note a computer would chose $r_1 \leftrightarrow r_3$ in this case to reduce rounding errors...

$$\begin{bmatrix} \boxed{1} & 2 & 8 & -1 & -2 & 5 \\ 0 & 1 & 2 & 0 & 3 & 10 \\ -2 & -3 & -14 & 3 & 8 & -2 \\ -3 & -4 & -20 & 4 & 15 & 9 \end{bmatrix}$$

$r_3 \leftarrow r_3 + 2r_1$

$r_4 \leftarrow r_4 + 3r$

$$\begin{bmatrix} 1 & 2 & 8 & -1 & -2 & 5 \\ 0 & \boxed{1} & 2 & 0 & 3 & 10 \\ 0 & 1 & 2 & 1 & 4 & 8 \\ 0 & 2 & 4 & 1 & 9 & 24 \end{bmatrix}$$

$r_3 \leftarrow r_3 - r_2$

$r_4 \leftarrow r_4 - 2r_2$

$$\begin{bmatrix} 1 & 2 & 8 & -1 & -2 & 5 \\ 0 & 1 & 2 & 0 & 3 & 10 \\ 0 & 0 & 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 1 & 3 & 4 \end{bmatrix}$$

$r_4 \leftarrow r_4 - r_3$

$$\begin{bmatrix} 1 & 2 & 8 & -1 & -2 & 5 \\ 0 & 1 & 2 & 0 & 3 & 10 \\ 0 & 0 & 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 0 & 2 & 6 \end{bmatrix} = U$$

$r_3 \leftarrow r_3 + 2r_1$
$r_4 \leftarrow r_4 + 3r_1$
$r_3 \leftarrow r_3 - r_2$
$r_4 \leftarrow r_4 - 2r_2$
$r_4 \leftarrow r_4 - r_3$

invert the effect of these row operations

$r_4 \leftarrow r_4 + r_3$
$r_4 \leftarrow r_4 + 2r_2$
$r_3 \leftarrow r_3 + r_2$
$r_4 \leftarrow r_4 - 3r_1$
$r_3 \leftarrow r_3 - 2r_1$

perform these operations on the identity to find L.

Matrix L collects the effect of these row operations.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -3 & 2 & 1 & 1 \end{bmatrix} = L$$

$$\begin{bmatrix} \boxed{1} & 2 & 8 & -1 & -2 & 5 \\ 0 & 1 & 2 & 0 & 3 & 10 \\ -2 & -3 & -14 & 3 & 8 & -2 \\ -3 & -4 & -20 & 4 & 15 & 9 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -3 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 8 & -1 & -2 & 5 \\ 0 & 1 & 2 & 0 & 3 & 10 \\ 0 & 0 & 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 0 & 2 & 6 \end{bmatrix}$$

What matrix corresponds to the row swap

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad r_1 \leftrightarrow r_2 \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = P$$

$$\begin{bmatrix} \boxed{0} & 1 & 2 & 0 & 3 & 10 \\ 1 & 2 & 8 & -1 & -2 & 5 \\ -2 & -3 & -14 & 3 & 8 & -2 \\ -3 & -4 & -20 & 4 & 15 & 9 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -3 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 8 & -1 & -2 & 5 \\ 0 & 1 & 2 & 0 & 3 & 10 \\ 0 & 0 & 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 0 & 2 & 6 \end{bmatrix}$$

$$\underset{4\times 6}{A} = \underset{4\times 4}{P}\ \underset{4\times 4}{L}\ \underset{4\times 6}{U}$$

U: (upper triangular) row echelon form

L: Lower triangular (elimination steps)

P: permutation matrix (row swaps)

Section 1.5

Solve $Ax=0$ same as $PLUx=0$

That can be solved in stages...

$$P(L(U(x)))=0$$

is equivalent to solving these systems in order...

$$\begin{cases} P(z)=0 \\ L(y)=z \\ U(x)=y \end{cases}$$

since P represents a row operation it is invertible. So the solution to $Pz=0$ is unique. Since $P0=0$ then $z=0$.

Similarly, $Ly=0$ and the fact that L corresponds to a sequence of invertible row operations means $y=0$

$Ux=0$ gives the same solutions as $Ax=0$

Make reduced row-echelon form:

$$U=\begin{bmatrix} \boxed{1} & 2 & 8 & -1 & -2 & 5 \\ 0 & \boxed{1} & 2 & 0 & 3 & 10 \\ 0 & 0 & 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 0 & \boxed{2} & 6 \end{bmatrix}$$

$r_1 \leftarrow r_1 - 2r_2$

pivots

$$\begin{bmatrix} \boxed{1} & 0 & 4 & -1 & -8 & -15 \\ 0 & \boxed{1} & 2 & 0 & 3 & 10 \\ 0 & 0 & 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 0 & \boxed{2} & 6 \end{bmatrix}$$

$r_1 \leftarrow r_1 + r_3$

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 0 & -7 & -17 \\ 0 & \boxed{1} & 2 & 0 & 3 & 10 \\ 0 & 0 & 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 0 & \boxed{2} & 6 \end{bmatrix}$$

$r_4 \leftarrow \frac{1}{2} r_4$

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 0 & -7 & -17 \\ 0 & \boxed{1} & 2 & 0 & 3 & 10 \\ 0 & 0 & 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$

$r_1 \leftarrow r_1 + 7r_4$

$r_2 \leftarrow r_2 - 3r_4$

$r_3 \leftarrow r_3 - r_4$

$$\begin{array}{r} 21 \\ -17 \\ \hline 4 \end{array}$$

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 0 & 0 & 4 \\ 0 & \boxed{1} & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & -5 \\ \underbrace{0 \quad 0 \quad 0 \quad 0}_{\text{zeros}} & & & & \boxed{1} & 3 \end{bmatrix} = R$$

reduced row echelon form of A

So solving $Ax=0$ is the same as $Ux=0$ which for the same reasons as before is the same as $Rx=0$

In terms of Augmented matrices

$$\left[A \,\middle|\, \begin{matrix}0\\0\\0\\0\end{matrix} \right] \xrightarrow[\text{operations}]{\text{row}} \left[U \,\middle|\, \begin{matrix}0\\0\\0\\0\end{matrix} \right] \xrightarrow[\text{operations}]{\text{row}} \left[R \,\middle|\, \begin{matrix}0\\0\\0\\0\end{matrix} \right]$$

$R x = 0$

$$\begin{array}{cccccc}
P & P & F & P & P & F \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6
\end{array}$$

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 0 & 0 & 4 \\ 0 & \boxed{1} & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & -5 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 + 4x_3 + 4x_6 = 0 \\ x_2 + 2x_3 + x_6 = 0 \\ x_4 - 5x_6 = 0 \\ x_5 + 3x_6 = 0 \end{cases}$$

solve is easy
just write the
pivot variables in
terms of the others

Solution

$$\begin{cases} x_1 = -4x_3 - 4x_6 \\ x_2 = -2x_3 - x_6 \\ x_4 = 5x_6 \\ x_5 = -3x_6 \end{cases}$$

In vector form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -4x_3 - 4x_6 \\ -2x_3 - x_6 \\ x_3 \\ 5x_6 \\ -3x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -4 \\ -1 \\ 0 \\ 5 \\ -3 \\ 1 \end{bmatrix} x_6$$

Solution to $Ax = 0$

The nullspace matrix.

$$N = \begin{bmatrix} -4 & -4 \\ -2 & -1 \\ 1 & 0 \\ 0 & 5 \\ 0 & -3 \\ 0 & 1 \end{bmatrix}$$