Systems of linear equation

$$Ax = b$$

Question, given A and b solve for x.

$$T(x) = b$$

Question, given T and b solve for x.

But first problem §1.5 #49 ...

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is $\boxed{\text{consistent}}$ for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

✓ ① If w is a solution so $Aw = b$ then $w = p + v_h$ where v_h is a solution so $Av_h = 0$.

② If v_h is a solution so $Av_h = 0$ then $w = p + v_h$ is a solution so $Aw = b$.

why is ① true... first note $Ap = b$

Suppose $Aw = b$. Subtract one equation from the other:

$$\cancel{\begin{array}{r} Ap = b \\ Aw = b \\ \hline Ap - Aw = 0 \end{array}}$$

$$\begin{array}{r} Aw = b \\ Ap = b \\ \hline Aw - Ap = 0 \end{array}$$

By linearity combine these terms...

$$A(w-p) = 0$$

Thus $p-w$ is a solution to the homogeneous equation. Thus $v_h = w-p$ then $Av_h = 0$.

Thus $w = p + v_h$.

2. If v_h is a solution so $Av_h = 0$ then $w = p + v_h$ is a solution so $Aw = b$.

Substitute it in

$$\boxed{Aw = A(p+v_h) = Ap + Av_h = b + 0 = b}.$$

49. Construct a 2×2 matrix A such that the solution set of the equation $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b}$ in $\mathbb{R}^2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$. Why does this *not* contradict Theorem 6?

$$\text{Nul}(A) = \{x : Ax = 0\} = \left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix} c : c \in \mathbb{R}\right\}$$

Row Echelon form of A

$$\begin{array}{cc} P & F \\ \end{array}$$
$$\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + a x_2 = 0$$
$$x_1 = -a x_2$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -a x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -a \\ 1 \end{bmatrix} x_2$$ so $a = -4$

For simplicity:

$$A = \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

Let

$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is inconsistent...

zero on the left non-zero on the right

No solutions, so not a line at all ...

Theorem 6 only applies when $Ax = b$ is consistent.

Back to Chapter 2.2

Systems of linear equations

$$Ax = b$$

Question, given A and b solve for x.

$$T(x) = b$$

Question, given T and b solve for x.

- does a solution exist?
- how many solutions?

Ideally for every b there is an x such that $Ax = b$ and only one such x such that $Ax = b$.

Invertible means there is a 1-to-1 correspondence that associates exactly one x to every b.

- need a pivot in every row so the system is never inconsistent
- there can be no free variables so there is only one solution..

If there is a pivot in each row, then there as as many pivots as there are rows.

If there are no free variables, then there are no extra columns.

Thus, there are the same number of columns as rows.

The matrix is square.

How to solve $T(x) = b$ in general...

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix}$$

Solve

$$\boxed{T(x) = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}, \quad T(y) = \begin{bmatrix} 0 \\ b_2 \\ 0 \end{bmatrix}, \quad T(z) = \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix}}$$

Then

$$T(x+y+z) = T(x) + T(y) + T(z).$$

$$= \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} = b$$

To solve this

$$T(x) = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}$$

First find w

$$\boxed{T(w) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

Then

Let $x = b_1 w$. Check $T(x) = \underbrace{T(b_1 w) = b_1 T(w)}_{\text{linear}} = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Suppose $T(u_1) = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $T(u_2) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $T(u_3) = \begin{bmatrix}0\\0\\1\end{bmatrix}$

Solve for u_1, u_2 and u_3 in these three problems...

Then to solve $T(x) = b = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

$$\boxed{x = b_1u_1 + b_2u_2 + b_3u_3 = \left[u_1 \middle| u_2 \middle| u_3\right]\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}}$$

$$T(x) = T(b_1u_1 + b_2u_2 + b_3u_3) = T(b_1u_1) + T(b_2u_2) + T(b_3u_3)$$

$$= b_1T(u_1) + b_2T(u_2) + b_3T(u_3)$$

$$= b_1\begin{bmatrix}1\\0\\0\end{bmatrix} + b_2\begin{bmatrix}0\\1\\0\end{bmatrix} + b_3\begin{bmatrix}0\\0\\1\end{bmatrix} = b.$$

Use the Augmented matrix to find u_1, u_2, u_3....

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$$A = \begin{bmatrix}1 & 0 & -2\\-3 & 1 & 4\\2 & -3 & 4\end{bmatrix}$$

with three right hand sides..

identity

$$\left[\begin{array}{ccc|c|c|c}1 & 0 & -2 & 1 & 0 & 0\\-3 & 1 & 4 & 0 & 1 & 0\\2 & -3 & 4 & 0 & 0 & 1\end{array}\right]$$

$r_2 \leftarrow r_2 + 3r_1$

$r_3 \leftarrow r_3 - 2r_1$

$$\left[\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\0 & 1 & -2 & 3 & 1 & 0\\0 & -3 & 8 & -2 & 0 & 1\end{array}\right]$$

$r_3 \leftarrow r_3 + 3r_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$

$r_1 \leftarrow r_1 + r_3$
$r_2 \leftarrow r_2 + r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$

$r_3 \leftarrow \frac{1}{2} r_3$

$$\begin{array}{c} \text{identity} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right] \\ \qquad\qquad\qquad\quad u_1 \quad u_2 \quad u_3 \end{array}$$

$$x = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} b \quad \text{solves} \quad \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} x = b.$$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

inverse of A

$$\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\leftarrow$ identity matrix so it works