Now solve

$$[A \mid e_1] \qquad [A \mid e_2] \qquad \text{and} \quad [A \mid e_3]$$

$$A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 3 & 2 \\ -2 & 6 & 5 \end{bmatrix}$$

Augmented matrix

identity matrix

pivot

$$\left[\begin{array}{ccc|c|c|c} 1 & -1 & 3 & 1 & 0 & 0 \\ -1 & 3 & 2 & 0 & 1 & 0 \\ -2 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} r_2 \leftarrow r_2 + r_1 \\ r_3 \leftarrow r_3 + 2r_1 \end{array}$$

pivot

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 4 & 11 & 2 & 0 & 1 \end{array}\right] \qquad r_3 \leftarrow r_3 - 2r_2$$

echelon form of A

$$U = \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \qquad \begin{array}{l} r_1 \leftarrow r_1 - 3r_3 \\ r_2 \leftarrow r_2 - 5r_3 \end{array}$$

Continue to make the reduced row echelon...

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 6 & -3 \\ 0 & 2 & 0 & 1 & 11 & -5 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \qquad r_1 \leftarrow r_1 + \frac{1}{2} r_2$$

$$\frac{12+11}{2} = \frac{23}{2}$$

$$-\frac{6+5}{2} = -\frac{11}{2}$$

rescale so pivots are all 1's.

$$r_2 \leftarrow \frac{1}{2} r_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 23/2 & -11/2 \\ 0 & \boxed{2} & 0 & 1 & 11 & -5 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

identity matrix

$$\begin{array}{cccccc} & & & u_1 & u_2 & u_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 23/2 & -11/2 \\ 0 & 1 & 0 & 1/2 & 11/2 & -5/2 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$u_1 = \begin{bmatrix} 3/2 \\ 1/2 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 23/2 \\ 11/2 \\ -2 \end{bmatrix} \quad \text{and} \quad u_3 = \begin{bmatrix} -11/2 \\ -5/2 \\ 1 \end{bmatrix}$$

$$Au_1 = e_1 \qquad Au_2 = e_2 \qquad \text{and} \qquad Au_3 = e_3$$

Suppose I want to solve $Ax = b$.

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1 e_1 + b_2 e_2 + b_3 e_3$$

$$= b_1 A u_1 + b_2 A u_2 + b_3 A u_3$$

$$= A b_1 u_1 + A b_2 u_2 + A b_3 u_3$$

$$b = A\underbrace{(b_1 u_1 + b_2 u_2 + b_3 u_3)}$$

If $x = b_1 u_1 + b_2 u_2 + b_3 u_3$ then $Ax = b$.

$$x = \left[\begin{array}{c|c|c} u_1 & u_2 & u_3 \end{array}\right] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3/2 & 23/2 & -11/2 \\ 1/2 & 11/2 & -5/2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

↗

Since this the matrix that finds x from b it's called the inverse matrix

If $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 3 & 2 \\ -2 & 6 & 5 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 3/2 & 23/2 & -11/2 \\ 1/2 & 11/2 & -5/2 \\ 0 & -2 & 1 \end{bmatrix}$

One more time: we're solving $Ax = b$.

mult both sides by A^{-1} $\quad \cancel{A^{-1}}\,\cancel{A}x = A^{-1}b$

Solution: $\quad x = A^{-1}b$

$A = \begin{bmatrix} 0 & 2 & 5 \\ 1 & -1 & 3 \\ -3 & 7 & 2 \end{bmatrix}$ want inverse matrix...

no pivot

swap

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 5 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ -3 & 7 & 2 & 0 & 0 & 1 \end{array}\right] \qquad r_1 \leftrightarrow r_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 0 \\ -3 & 7 & 2 & 0 & 0 & 1 \end{array}\right] \qquad r_3 \leftarrow r_3 + 3r_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 0 \\ 0 & 4 & 11 & 0 & 3 & 1 \end{array}\right] \qquad r_3 \leftarrow r_3 - 2r_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 3 & 1 \end{array}\right]$$

$r_1 \leftarrow r_1 - 3r_3$
$r_2 \leftarrow r_2 - 5r_3$

Nowreduced row echelon Form

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 6 & -8 & -3 \\ 0 & 2 & 0 & 11 & -15 & -5 \\ 0 & 0 & 1 & -2 & 3 & 1 \end{array}\right]$$

$r_1 \leftarrow r_1 + \frac{1}{2} r_2$

$$\frac{-16}{2} - \frac{15}{2} = \frac{-31}{2}$$

$$-\frac{6}{2} - \frac{5}{2} = \frac{-11}{2}$$

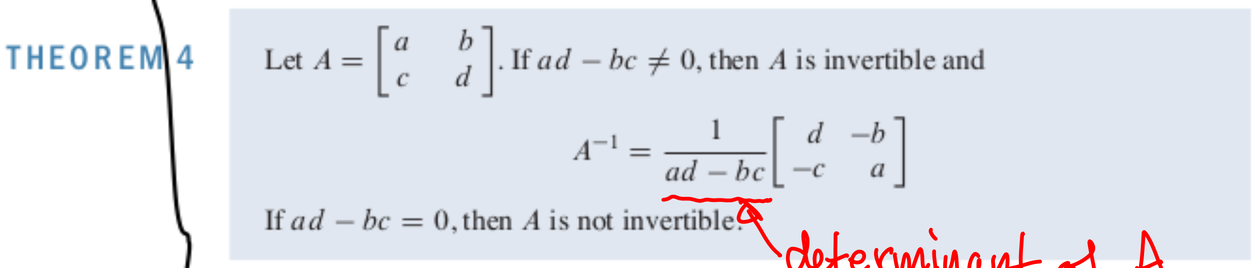
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 23/2 & -31/2 & -11/2 \\ 0 & 2 & 0 & 11 & -15 & -5 \\ 0 & 0 & 1 & -2 & 3 & 1 \end{array}\right]$$

rescale

$r_2 \leftarrow \frac{1}{2} r_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 23/2 & -31/2 & -11/2 \\ 0 & 1 & 0 & 11/2 & -15/2 & -5/2 \\ 0 & 0 & 1 & -2 & 3 & 1 \end{array}\right] \leftarrow A^{-1}$$

$$A^{-1} = \begin{bmatrix} 23/2 & -31/2 & -11/2 \\ 11/2 & -15/2 & -5/2 \\ -2 & 3 & 1 \end{bmatrix}$$



THEOREM 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

determinant of A

$A = \begin{bmatrix} 7 & 2 \\ 6 & 1 \end{bmatrix}$ $a = 7$ $b = 2$ $c = 6$ $d = 1$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{7\cdot 1 - 2\cdot 6}\begin{bmatrix} 1 & -2 \\ -6 & 7 \end{bmatrix}$$

$$= \frac{-1}{5}\begin{bmatrix} 1 & -2 \\ -6 & 7 \end{bmatrix} = \begin{bmatrix} -1/5 & 2/5 \\ 6/5 & -7/5 \end{bmatrix}$$

$$\begin{bmatrix} -1/5 & 2/5 \\ 6/5 & -7/5 \end{bmatrix}$$

$$7\cdot\left(-\frac{1}{5}\right) + 2\left(\frac{6}{5}\right) = \frac{5}{5} = 1$$

$$6\left(\frac{2}{5}\right) + 1\left(-\frac{7}{5}\right) = \frac{5}{5} = 1$$

$$\begin{bmatrix} 7 & 2 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Identity matrix

$$\begin{bmatrix} 7 & 2 \\ 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1/5 & 2/5 \\ 6/5 & -7/5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Identity matrix

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

Transposes: $(AB)^T = B^T A^T$

$AA^{-1} = I$

definition of A^{-1}

$(AA^{-1})^T = I^T$

$(A^{-1})^T A^T = I$

this is a matrix that multiplies A^T to get I

$A^{-1}A = I$

$(A^{-1}A)^T = I^T$

$A^T (A^{-1})^T = I$

this is a matrix that multiplies A^T to get I

$\Leftrightarrow$

These two things together mean the inverse of A^T

$$(A^T)^{-1} = (A^{-1})^T$$