Theorem:

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$

then $\det AB = \det A \det B$.

(save this for Cramer's rule)

① Suppose $\det A \neq 0$. Then A is not invertible...

☑ ② Suppose A is not invertible. Then $\det A = 0$.

Explain ② first; Suppose A is not invertible...

algorithm for finding an inverse:

$$[A \mid I] \xrightarrow{\text{Gaussian Elimination}} [I \mid A^{-1}]$$

If not invertible something goes wrong with the elimination and there are not enough pivots...

Do row operations: Elimination steps
Row swaps...

... but run out of pivots, which means one of the rows is the zero row...

Echelon form of A looks like

$$U = \begin{bmatrix} * & & & & \\ 0 & * & & & \\ \vdots & 0 & & * & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

row of zeros at the bottom...

$\det A = \pm \det U$

expand $\det U$ along the bottom row of zeros...

$$\det U = u_{n1}C_{n1} + u_{n2}C_{n2} + \cdots + u_{nn}C_{nn} = 0$$

where $C_{ij} = (-1)^{i+j} \det U_{ij}$

since all the entries in the last row of U are zero...

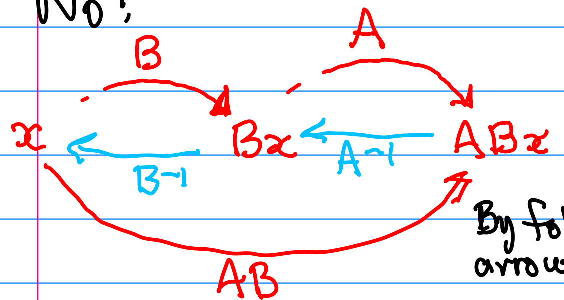
Case when AB is invertible

☑ Case when AB is not invertible...

If AB is not invertible then $\boxed{\det AB = 0}$.

Q: Is it possible that both A and B on their own are invertible even though AB is not?

A: No!



By following the arrows:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$AB\,B^{-1}A^{-1} = AA^{-1} = I$$

Aside: It's possible to have two matrices which are not invertible, but for which AB is invertible.

This can happen when A and B are not square so $\det A$ doesn't make sense

Therefore either A or B must not be invertible since AB was not invertible...

If A is not invertible then $\det A = 0$

If B is not invertible then $\det B = 0$

Either way $(\det A)(\det B) = 0$.

Recall,

$\det AB = 0$. Therefore $\det AB = \det A \det B$

Suppose AB is invertible and A and B are square... It follows that A and B must be invertible...

AB invertible means $(AB)^{-1}$ exists

why

Claim: $A^{-1} = B(AB)^{-1}$

Check $A[B(AB)^{-1}] = AB(AB)^{-1} = I$

Claim: $B^{-1} = \boxed{\begin{matrix} \cancel{A(AB)^{-1}} \\ (AB)^{-1}A \end{matrix}}$

consistency, if A^{-1} and B^{-1} exist then

$A(AB)^{-1} = AB^{-1}A^{-1}$

and

$(AB)^{-1}A = B^{-1}A^{-1}A$

Check $[(AB)^{-1}A]B = (AB)^{-1}AB = I$

Now A and B are invertible... means they can be reduced using row operations to the identity.

Thus A can be obtained from the identity matrix by performing a sequence of elementary row operations.

$$\boxed{A = \underbrace{E_1 E_2 E_3 \cdots E_n}}$$

all these E_i's correspond to elementary row operations

why is $\det AB = \det A \det B$?

Easier Question:

☑ Why is $\det EB = \det E \det B$ when E is an elementary matrix? Because of the theorem

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

each one of these properties says that a particular type of row operation can be factored out of the determinant.

Consider $r_1 \leftrightarrow r_2$ what's the elemenary matrix for that?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad r_1 \leftrightarrow r_2 \quad E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det I = 1 \qquad \det E = -\det I = -1$$

$$\det EA = -1 \det A = \det E \det A \quad \checkmark$$

In summary we have

1. $A = \underbrace{E_1 E_2 E_3 \cdots E_n}$

2. $\det EB = \det E \det B$

$$\det AB = \det (E_1)(E_2 E_3 \cdots E_n B)$$
$$= \det E_1 \det (E_2)(E_3 \cdots E_n B)$$
$$= \det E_1 \det E_2 \det (E_3 \cdots E_n B)$$
$$\vdots$$
$$= (\det E_1)(\det E_2) \cdots (\det E_n)(\det B)$$
$$= (\det E_1)(\det E_2) \cdots (\det E_{n-1} E_n)(\det B)$$
$$\vdots$$
$$= (\det E_1)(\det E_2 \cdots E_n)(\det B)$$
$$= \det (E_1 E_2 \cdots E_n) \det (B) = \det A \det B.$$

Thus, $\det AB = \det A \det B$.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bd.$$

$$\det \begin{bmatrix} e & f \\ g & h \end{bmatrix} = eh - fg.$$

$$\begin{array}{cc} & \begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \begin{bmatrix} ae+bg & af+bh \\ ce+dg & ef+dh \end{bmatrix} \end{array}$$

$$\det \begin{bmatrix} ae+bg & af+bh \\ ce+dg & ef+dh \end{bmatrix} = (ae+bg)(ef+dh) - (af+bh)(ce+dg)$$

"difficult algebra" $= (ad-bd)(eh-fg) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \det \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$

This makes the explanation of $\det AB = \det A \det B$ using Gaussian elimination look relatively simple.