The eigenvalue eigenvector program transforms matrix vector multiplication
into scalar vector multiplication...that's much simpler...

If x is an eigenvector for $A \in \mathbb{R}^{n \times n}$

and λ is the corresponding eigenvalue,

a scalar...

This is n equations with n+1 unknowns

Then

$$\boxed{Ax = \lambda x}$$

This is a quadratic eq. if you are solving for both λ and x.

Idea given A then solve for x and λ

$$Ax - \lambda x = 0$$

$$\underbrace{(A - \lambda I)}_{B = A - \lambda I} x = 0$$

exclude the solution $x=0$ because it's not useful.

Solving $Bx=0$ (homogeneous equation) the x's which satisfy this $Nul(B) = \{x : Bx = 0\}$.

B must have free variable for there to be a non-zero solution x to $Bx=0$.

For B to have free variables means B is **not** invertible

If B is not invertible then $\det B = 0$.

(if $\det B \neq 0$ then Cramer's rule would imply B was invertible...)

$\boxed{B = A - \lambda I}$

Need $\det(A - \lambda I) = 0$

Called characteristic equation.

Characteristic polynomial $\chi(\lambda) = \det(A - \lambda I)$.

Thus $\chi(\lambda) = 0$ is the characteristic equation.

Example:

13. $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $\lambda = 1, 2, 3$

Trying to solve $\chi(\lambda) = 0$ i.e. verify that $\chi(1) = 0$, $\chi(2) = 0$ and $\chi(3) = 0$

$$\chi(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}$$

difficult to use Gaussian elimination to find the determinant, because the variable λ is unknown so it's difficult to know what terms are zero.

Note setting $\lambda = 1$ yields

$$\chi(1) = \det \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = 0$$

two rows are the same means not invertible so the determinant is zero.

$$\chi(\lambda) = \det \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}$$

B

$$= \underbrace{-0 \det B_{12}}_{1+2 \text{ odd}} \underbrace{+(1-\lambda)\det B_{22}}_{2+2 \text{ even}} \underbrace{-0 \det B_{32}}_{3+2 \text{ odd}}$$

$$= (1-\lambda)\det \begin{bmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix} = (1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right]$$

$$= (1-\lambda)\left[4-5\lambda+\lambda^2+2\right] = (1-\lambda)(\lambda^2-5\lambda+6)$$

$$= (1-\lambda)(\lambda-3)(\lambda-2) = 0 \qquad \text{yield } \lambda = 1, 2 \text{ or } 3.$$

this is a cubic eq. for λ,

GOAL

Use this algorithm to factor the matrix A

First let's solve for the $\underline{\underline{x}}$ vectors...

there will be lots of them...

$\lambda = 1$

$$\mathrm{Nul}(A - 1\cdot I) = \mathrm{Nul}\begin{bmatrix} 4-1 & 0 & 1 \\ -2 & 1-1 & 0 \\ -2 & 0 & 1-1 \end{bmatrix}$$

$$= \mathrm{Nul}\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \left\{ x : \begin{array}{c} \quad F \quad \\ \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\}$$

vector form of solution...

$$\begin{cases} 3x_1 + x_3 = 0 \\ -2x_1 = 0 \\ -2x_1 = 0 \end{cases} \qquad \begin{array}{l} x_1 = 0 \\ x_3 = 0 \\ x_2 = x_2 \end{array} \qquad x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2$$

Choose $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

eigenvector (not unique)

any of these x's are eigenvectors for $\lambda = 1$

$\lambda = 2$

$$\mathrm{Nul}(A - 2 \cdot I) = \mathrm{Nul}\begin{bmatrix} 4-2 & 0 & 1 \\ -2 & 1-2 & 0 \\ -2 & 0 & 1-2 \end{bmatrix}$$

$$= \mathrm{Nul}\underbrace{\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}}_{B}$$

Solve $Bx = 0$

$r_2 \leftarrow r_2 + r_1$

$r_3 \leftarrow r_3 + r_1$

$$= \mathrm{Nul}\ \begin{array}{c} \begin{matrix} P & P & F \end{matrix} \\ \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$2x_1 + x_3 = 0$ $\quad x_1 = -\frac{1}{2}x_3$

$-x_2 + x_3 = 0$ $\quad x_2 = x_3$

vector form solution to $Bx = 0$

$$x = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} x_3$$

eigenvector $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$\lambda = 3$.

$$\mathrm{Nul}(A - 3I) = \mathrm{Nul}\begin{bmatrix} 4-3 & 0 & 1 \\ -2 & 1-3 & 0 \\ -2 & 0 & 1-3 \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

$r_2 \leftarrow r_2 + 2r_1$
$r_3 \leftarrow r_3 + 2r_1$

$$= \text{Nul}\begin{array}{c} \begin{array}{ccc} P & P & F \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$r_2 \leftarrow -\frac{1}{2} r_2$

$$= \text{Nul}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_3 = 0$
$x_2 - x_3 = 0$

$x_1 = -x_3$
$x_2 = x_3$
$x_3 = \text{free}$

Vector form solution is

$$x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$$

Corresponding eigenvector to $\lambda = 3$ is $x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

In summary

eigenvalue	eigenvector
$\lambda = 1$	$x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
$\lambda = 2$	$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$
$\lambda = 3$	$= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Use this to factor the matrix A.

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \left[\, 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; 2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \;\middle|\; 3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \,\right]$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Let.

$$S = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

then

$AS = SD$ thus $\boxed{A = SDS^{-1}}$ ← factorization

mult on the right by S^{-1}