Complex numbers:

when we solve

$$\chi(\lambda) = 0$$

$$\det(A - \lambda I) = 0$$

then λ might be complex.

$A \in \mathbb{R}^{n\times n}$

$\lambda \in \mathbb{C}$

set of all complex numbers

x is the eigenvector

this side real

$$\overbrace{Ax}^{} = \overbrace{\lambda x}^{}$$

then this would be complex

real

suppose x is real

complex

suppose x is real

If $\lambda = a + ib$ where $a, b \in \mathbb{R}$ and $b \neq 0$

could it happen that $x \in \mathbb{R}^n$? (Yes or No)

Therefore $x \in \mathbb{C}^n$ and $x \notin \mathbb{R}^n$

a vector with complex entries...

Suppose $A \in \mathbb{R}^{2\times 2}$ and $\lambda \in \mathbb{C}$

with $\lambda = a + ib$ where $a, b \in \mathbb{R}$ and $b \neq 0$.

Then $x \in \mathbb{C}^2$ and $x = u + iv$ where $u, v \in \mathbb{R}^2$.

$$x = \begin{bmatrix} 2-i \\ 3+4i \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + i \begin{bmatrix} -1 \\ 4 \end{bmatrix} = u + iv, \quad u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} 2+i \\ 3-4i \end{bmatrix} = u - iv$$

$$Ax = A(u+iv) = Au + iAv$$

equal

also $\quad \Vert$

$$\lambda x = (a+ib)(u+iv) = au + iav + ibu + i^2 bv$$

$$= au - bv + i(av+bu)$$

Thus

$$Au = au - bv \quad \text{and} \quad Av = av + bu$$

combine these two linear systems into one matrix equation

column operation

$$A\left[u \mid v\right] = \left[Au \mid Av\right] = \left[au-bv \mid av+bu\right]$$

$$= \left[u \mid v\right]\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

factor out the a's and b's ..

Let $P = \left[u \mid v\right]$ then $A = P\begin{bmatrix} a & b \\ -b & a \end{bmatrix}P^{-1}$

Recall $\lambda = a+ib$ and $x = u+iv$ and $Ax = \lambda x$

$$\overline{Ax} = \overline{\lambda x}$$

since A is real $\bar{A} = A$

Thus $A\bar{x} = \overline{\lambda x} = \bar{\lambda}\bar{x}$

thus $\bar{x}$ is an eigenvector with eigenvalue $\bar{\lambda}$

Two eigenvalues and two eigenvectors...

$$Ax=\lambda x \quad \text{and} \quad A\bar{x}=\bar{\lambda}\bar{x}$$

Diagonalize A by taking

$$S=\left[\, x \mid \bar{x} \,\right] \qquad D=\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}$$

so that $$A = S\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}S^{-1}$$

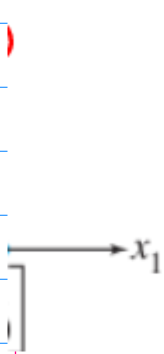
$$A = P\underbrace{\begin{bmatrix}a & b\\ -b & a\end{bmatrix}}_{\text{rotation matrix}}P^{-1}$$

compare two factorization and there a tradeoffs
diagonal versus real values...

$a^2+b^2=c^2$ might not be 1.

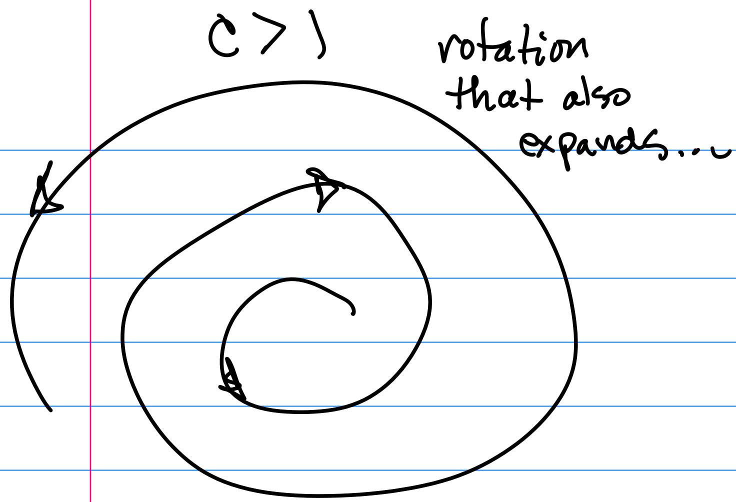
From the Geometry of Matrices Lecture

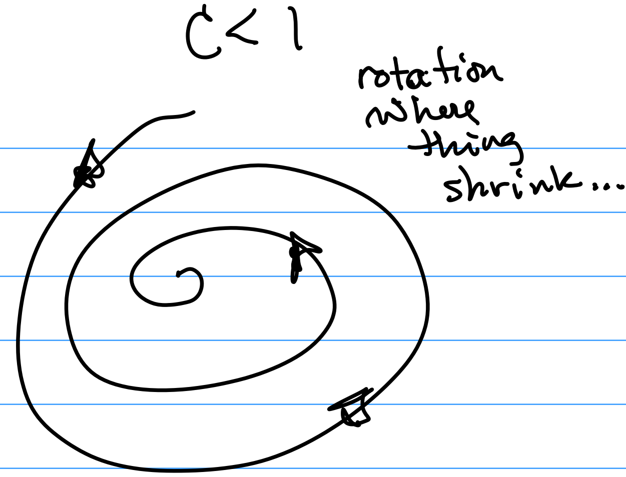
rotation θ degrees counter clockwise



$$R_\theta=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$\cos^2\theta+\sin^2\theta=1$$





$$A = P \begin{bmatrix} a & b \\ -b & a \end{bmatrix} P^{-1} \qquad R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$a^2 + b^2 = c^2$$

Then

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = cR_\theta$$

solve for θ using trigonometry s_s

$$\begin{cases} c\cos\theta = a \\ c\sin\theta = -b \end{cases}$$

$$\boxed{\text{If } A^T = A \text{ and } A \in \mathbb{R}^{n\times n} \text{ then } \lambda \text{ is guaranteed to be real.}}$$

part of the Spectral theorem...

The spectrum of A is the set of eigenvalues of A.
The spectral theorem says if $A = A^T$ then λ is real.

Exercises from 5.5

23. Let A be an $n \times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q = \overline{\mathbf{x}}^T A\mathbf{x}$. The equalities below show that q is a real number by verifying that $\overline{q} = q$. Give a reason for each step.

$$\overline{q} \underset{\text{(a)}}{=} \overline{\overline{\mathbf{x}}^T A\mathbf{x}} \underset{\text{(b)}}{=} \mathbf{x}^T \overline{A\mathbf{x}} \underset{\text{(c)}}{=} \mathbf{x}^T A\overline{\mathbf{x}} \underset{\text{(d)}}{=} (\mathbf{x}^T A\overline{\mathbf{x}})^T \underset{\text{(e)}}{=} \overline{\mathbf{x}}^T A^T \mathbf{x} = q$$

24. Let A be an $n \times n$ real matrix with the property that $A^T = A$. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$ in $\mathbb{C}^n$, then, in fact, λ is real and the real part of $\mathbf{x}$ is an eigenvector of A. [*Hint:* Compute $\overline{\mathbf{x}}^T A\mathbf{x}$, and use Exercise 23. Also, examine the real and imaginary parts of $A\mathbf{x}$.]

Spectral theorem

Theorem about the eigenvalues...

definition of matrix mult.

$$\bar{x} \cdot Ax = Ax \cdot \bar{x} = (Ax)^T \bar{x} = x^T A^T \bar{x}$$

$$= x^T A\bar{x} = x \cdot A\bar{x}$$

Therefore

$$\bar{x} \cdot Ax = A\bar{x} \cdot x$$

recall

Two eigenvalues and two eigenvectors...

$$Ax = \lambda x \quad \text{and} \quad A\bar{x} = \bar{\lambda}\bar{x}$$

Therefore

$$\bar{x} \cdot \lambda x = \bar{\lambda}\bar{x} \cdot x$$

$$\lambda \cancel{\bar{x} \cdot x} = \bar{\lambda} \cancel{\bar{x} \cdot x}$$

$\lambda = \bar{\lambda}$ which means λ is real

Can't cancel zero, so need to make sure $\bar{x} \cdot x \neq 0$.

$$0 = 0$$
$$2 \cdot \cancel{0} = 3 \cdot \cancel{0}$$
$$2 = 3 \quad \text{nonsense} \ldots$$

$x \neq 0, \quad x \in \mathbb{C}^n$

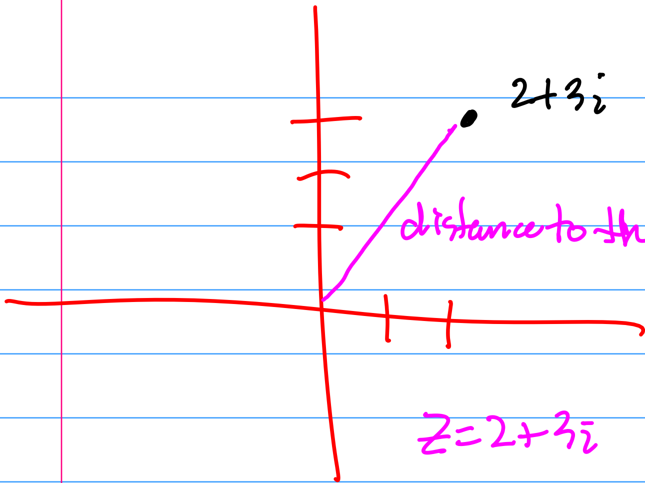
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

$$\bar{x} \cdot x = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\bar{x}_1 x_1 + \bar{x}_2 x_2 + \cdots + \bar{x}_n x_n}_{\text{all these terms are } \geq 0 \text{ and at least one of them is } > 0} > 0$$

complex number mult by it's conjugate

$$(2-3i)(2+3i) = 4 + \cancel{6i} - \cancel{6i} - 9i^2 = \underset{2^2}{4} + \underset{3^2}{9} = 13$$

$$(a+ib)(a-ib) = a^2 + b^2 > 0$$



distance to the origin = $\sqrt{2^2+3^2}$

absolute value or modulus of a complex number

$|2+3i| = \sqrt{13}$

$z = 2+3i$ then $|z| = \sqrt{\bar{z}\, z}$

$\|x\| = \sqrt{\bar{x} \cdot x}$