Gram-Schmidt Algorithm.

$$u_1 = x_1 \qquad x_1 = u_1 = \|u_1\| v_1 \qquad v_1 = \frac{u_1}{\|u_1\|}$$

$$u_2 = x_2 - (v_1 \cdot x_2) v_1 \qquad v_2 = \frac{u_2}{\|u_2\|}$$

note, even though x_2 appears here, I know what these dot products are because I just did Gram-Schmidt.

$$x_2 = (v_1 \cdot x_2) v_1 + u_2 = (v_1 \cdot x_2) v_1 + \|u_2\| v_2$$

$$u_3 = x_3 - (v_1 \cdot x_3) v_1 - (v_2 \cdot x_3) v_2 \qquad v_3 = \frac{u_3}{\|u_3\|}$$

$$x_3 = (v_1 \cdot x_3) v_1 + (v_2 \cdot x_3) v + u_3 = (v_1 \cdot x_3) v_1 + (v_2 \cdot x_3) v + \|u_3\| v_3$$

$$u_n = x_n - (v_1 \cdot x_n) v_1 - \cdots - (v_{n-1} \cdot x_n) v_{n-1} \qquad v_n = \frac{u_n}{\|u_n\|}$$

Recall

$$A = \left[x_1 \,\middle|\, x_2 \,\middle|\, \cdots \,\middle|\, x_n \right] \quad \text{and} \quad Q = \left[v_1 \,\middle|\, v_2 \,\middle|\, \cdots \,\middle|\, v_n \right]$$

row, column

$$R = \begin{bmatrix} \|u_1\| & v_1 \cdot x_2 & v_1 \cdot x_3 & \cdots & v_1 \cdot x_n \\ 0 & \|u_2\| & v_2 \cdot x_3 & & v_2 \cdot x_n \\ 0 & 0 & \|u_3\| & & v_3 \cdot x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & v_{n-1} \cdot x_n \\ 0 & 0 & 0 & & \|u_n\| \end{bmatrix}$$

Mult by this matrix on the right undoes what Gram-Schmidt did in order to make Q. Thus

$$A = QR.$$

$$\mathbf{9.}\quad A = \begin{array}{ccc} x_1 & x_2 & x_3 \\ \shortparallel & \shortparallel & \shortparallel \end{array}$$
$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

$$u_1 = x_1 \qquad v_1 = \frac{u_1}{\|u_1\|}$$

$$u_2 = x_2 - (v_1 \cdot x_2)\, v_1 \qquad v_2 = \frac{u_2}{\|u_2\|}$$

$$u_3 = x_3 - (v_1 \cdot x_3)\, v_1 - (v_2 \cdot x_3)\, v_2 \qquad v_3 = u_3 / \|u_3\|$$

$$Q = \begin{bmatrix} v_1 \,|\, v_2 \,|\, v_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2\sqrt5} & \frac{1}{2\sqrt5} & \frac{-3}{2\sqrt5} \\ \frac{1}{2\sqrt5} & \frac{3}{2\sqrt5} & \frac{1}{2\sqrt5} \\ \frac{-1}{2\sqrt5} & \frac{3}{2\sqrt5} & \frac{1}{2\sqrt5} \\ \frac{3}{2\sqrt5} & \frac{-1}{2\sqrt5} & \frac{3}{2\sqrt5} \end{bmatrix} \text{ and } R = \begin{bmatrix} \|u_1\| & v_1 \cdot x_2 & v_1 \cdot x_3 \\ 0 & \|u_2\| & v_2 \cdot x_3 \\ 0 & 0 & \|u_3\| \end{bmatrix} = \begin{bmatrix} 2\sqrt5 & -4\sqrt5 & 3\sqrt5 \\ 0 & 2\sqrt5 & -\sqrt5 \\ 0 & 0 & 2\sqrt5 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \qquad v_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{9+1+1+9}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{2\sqrt5} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

do the dot product first so we know what goes in R,

$$u_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \left(\frac{1}{2\sqrt5} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} \right) \frac{1}{2\sqrt5} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \left(\frac{-15+1-5-21}{2\sqrt5} \right) \frac{1}{2\sqrt5} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \left(\frac{-20}{\sqrt5} \right) \frac{1}{2\sqrt5} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \left(-4\sqrt5 \right) \frac{1}{2\sqrt5} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} \qquad v_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{1+9+9+1}} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt5} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$u_3 = x_3 - (v_1 \cdot x_3)\, v_1 - (v_2 \cdot x_3)\, v_2$$

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \left(\frac{1}{2\sqrt5} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \right) \frac{1}{2\sqrt5} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \left(\frac{1}{2\sqrt5} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \right) \frac{1}{2\sqrt5} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \left(\frac{1}{2\sqrt5} \underbrace{(3+1+2+24)}_{30} \right) \frac{1}{2\sqrt5} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \left(\frac{1}{2\sqrt5} \underbrace{(1+3-6-8)}_{-10} \right) \frac{1}{2\sqrt5} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - (3\sqrt5) \frac{1}{2\sqrt5} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - (-\sqrt5) \frac{1}{2\sqrt5} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\frac{30}{2\sqrt5} = \frac{15}{\sqrt5} \cdot \frac{\sqrt5}{\sqrt5} = 3\sqrt5$$

$$= \begin{bmatrix}1\\1\\-2\\8\end{bmatrix} - \frac{3}{2}\begin{bmatrix}3\\1\\-1\\3\end{bmatrix} + \frac{1}{2}\cdot\begin{bmatrix}1\\3\\3\\-1\end{bmatrix} = \begin{bmatrix}-3\\1\\1\\3\end{bmatrix}$$

$$8 - \frac{9}{2} - \frac{1}{2} = 8 - 5 = 3$$

$$v_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{9+1+1+9}}\begin{bmatrix}-3\\1\\1\\3\end{bmatrix} = \frac{1}{2\sqrt{5}}\begin{bmatrix}-3\\1\\1\\3\end{bmatrix} = \begin{bmatrix}\frac{-3}{2\sqrt5}\\ \frac{1}{2\sqrt5}\\ \frac{1}{2\sqrt5}\\ \frac{3}{2\sqrt5}\end{bmatrix}$$

Therefore $A = QR$

$$\begin{bmatrix}3 & -5 & 1\\ 1 & 1 & 1\\ -1 & 5 & -2\\ 3 & -7 & 8\end{bmatrix} = \begin{bmatrix}\frac{3}{2\sqrt5} & \frac{1}{2\sqrt5} & \frac{-3}{2\sqrt5}\\ \frac{1}{2\sqrt5} & \frac{3}{2\sqrt5} & \frac{1}{2\sqrt5}\\ \frac{-1}{2\sqrt5} & \frac{3}{2\sqrt5} & \frac{1}{2\sqrt5}\\ \frac{3}{2\sqrt5} & \frac{-1}{2\sqrt5} & \frac{3}{2\sqrt5}\end{bmatrix}\begin{bmatrix}2\sqrt5 & -4\sqrt5 & 3\sqrt5\\ 0 & 2\sqrt5 & -\sqrt5\\ 0 & 0 & 2\sqrt5\end{bmatrix}$$

$$\frac{1}{2\sqrt5}\begin{bmatrix}1\\3\\1\\ \cdot\end{bmatrix}\cdot\sqrt5\begin{bmatrix}-4\\2\\0\end{bmatrix} = \frac{1}{2}(-4+6+0) = \frac{2}{2} = 1$$

Example

$$Ax = b \qquad A = QR$$

upper triangular

satisfies $Q^TQ = I$ use this

Has orthonormal columns

$$QRx = b$$

Could solve this as two systems like before

$$Qy = b$$
$$Rx = y$$

$$Q^TQRx = Q^Tb$$

$$Rx = Q^Tb$$

matrix vector multiplication is even easier...

triangular so can solve for x using substitution