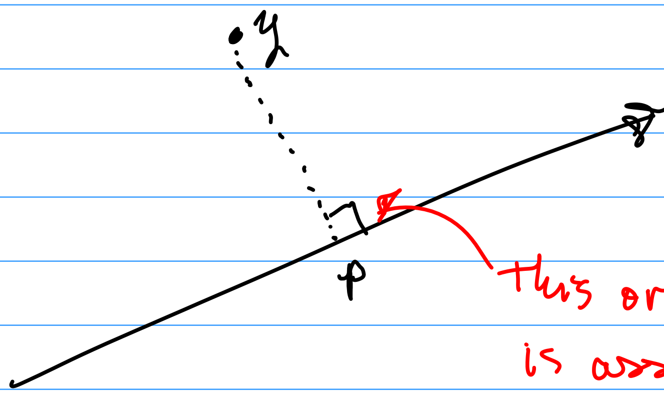
about orthogonal projections...



- find point on the line closest to y

this orthogonality (right angle) is associated with the optimality of p being closest to the point y.

Orthogonality has something to do with dot products...

Transposes have something to do with dot products...

$$y \cdot Ax = Ax \cdot y = (Ax)^T y = x^T A^T y$$
$$= x^T (A^T y) = x \cdot A^T y = A^T y \cdot x$$

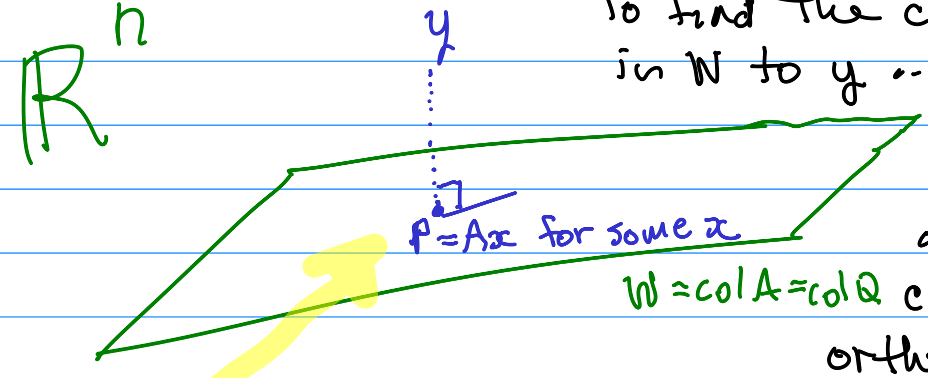
Thus

the A jumps over the dot and gets transposed...

$$y \cdot Ax = A^T y \cdot x$$

<u>Goal</u> find the closest point in a plane or hypersurface (actually a subspace) to another point...

Let W be a subspace of $\mathbb{R}^n$



To find the closest point p in W to y ... need this orthogonality.

This is called the orthogonal projection of y onto W.

To find p...

How: Let $\{b_1, b_2, \ldots, b_q\} \subseteq \mathbb{R}^n$ be a basis for W.

What does that mean:

① The span of the b_i's is W...

② The b_i's are linearly independent...

Describe ① and ② using matrices

$$A = \begin{bmatrix} b_1 & b_2 & \cdots & b_q \end{bmatrix} \in \mathbb{R}^{n \times q}$$

$$\text{Col}\,A = \{Ax : x \in \mathbb{R}\} = \text{span}\{b_1, b_2, \ldots, b_q\} = W$$

Note the b's being linearly independent means

$$\text{Nul}\,A = \{x : Ax = 0\} = \{0\} \text{ thus } \dim \text{Nul}\,A = 0$$

Since the columns of A are linearly independent, then

$A = QR$ by Gram-Schmidt algorithm.

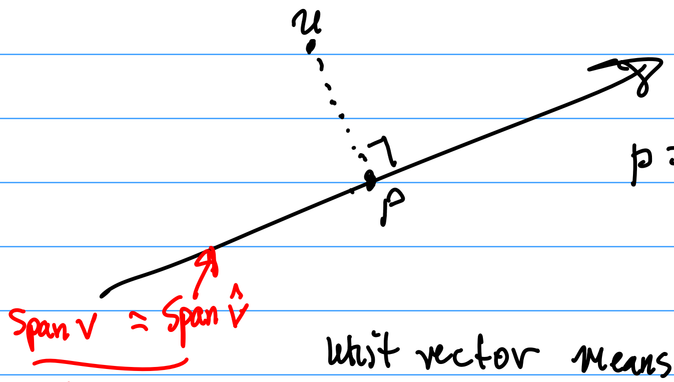
from last week...

Then the distance of the projection from the origin is in the v direction

$$\|p\| = u \cdot \hat{v} = (2,3)\cdot\frac{(4,1)}{\sqrt{17}} = \frac{8+3}{\sqrt{17}} = \frac{11}{\sqrt{17}}$$

Thus

$$p = \|p\|\,\hat{v} = \frac{11}{\sqrt{17}}\cdot\frac{(4,1)}{\sqrt{17}} = \frac{11}{17}(4,1) = \left(\frac{44}{17}, \frac{11}{17}\right)$$



$$p = \|p\|\hat{v} = (u\cdot\hat{v})\,\hat{v}$$

unit vector means $\|\hat{v}\| = 1$ or $\|\hat{v}\|^2 = 1$

$$\hat{v}\cdot\hat{v} = 1 \quad \text{or} \quad \hat{v}^T\hat{v} = 1$$

Compare this with $Q^TQ = I$

Use this to find p in a higher dimensional space W.

Note that Col A = Col Q. Why?

The Gram-Schmidt process uses column operations on the columns of A to obtain the columns of Q. Since the column operations are invertible, the columns of A can also be obtained from the columns of Q. Therefore Col A = Col Q.

Correct but there is a more general relationship

$$\underset{n\times q}{A} = \underset{n\times q}{Q}\ \underset{q\times q}{R}$$

$$\text{Col}\,A = \{Ax : x \in \mathbb{R}^q\} = \{QRx : x \in \mathbb{R}^q\}$$

$y = Rx \in \mathbb{R}^q$ $\qquad$ $\{Rx : x \in \mathbb{R}^q\} \subseteq \mathbb{R}^q$

$$\boxed{\operatorname{col} A \subseteq \{Qy : y \in \mathbb{R}^q\} = \operatorname{col} Q}$$

more general

No matter what factorization $A = BC$

$$\operatorname{col} A \subseteq \operatorname{col} B$$

Know R is invertible so $A = QR$ then $\boxed{Q = AR^{-1}}$

this is a factorization of Q

$$\boxed{\operatorname{col} Q \subseteq \operatorname{col} A}$$

Since $\operatorname{col} A \subseteq \operatorname{col} Q$ and $\operatorname{col} Q \subseteq \operatorname{col} A$ then it must be that $\operatorname{col} A = \operatorname{col} Q$...

What is p?

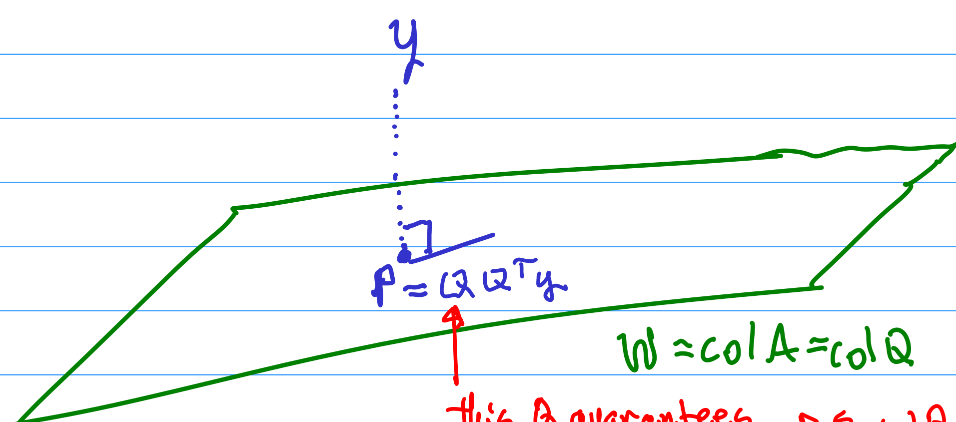
$$p = \|p\| \hat{v} = (u \cdot \hat{v}) \hat{v} = \hat{v} (\hat{v}^T u)$$

1-dimensional case..

n-dimensional case...

$$p = Q Q^T y$$

Check this...



- To check $p = QQ^Ty$ I need to show the vector from y to p is perpendicular to any vector $w \in W$.

this vector is $p-y$

$w \in \text{col}\,Q$ so $w = Qz$ for some $z \in \mathbb{R}^q$

To be perpendicular the dot product must be zero

$$(p-y)\cdot w = p\cdot w - y\cdot w = QQ^Ty\cdot w - y\cdot w$$

$$= QQ^Ty\cdot Qz - y\cdot Qz$$

try to get rid of some of the Q's here using the fact that $Q^TQ = I$

$$= Q^TQQ^Ty\cdot z - Q^Ty\cdot z$$

$$= Q^Ty\cdot z - Q^Ty\cdot z = 0$$

recall fact about transposes...

$$y\cdot Ax = A^Ty\cdot x$$

In summary to find the orthogonal projection of y onto W or the closest point in W to y...

Let $p = QQ^Ty$

where $A = QR$ is the QR factorization of A and $A = [b_1 | b_2 | \dots b_q]$ with $\{b_1, b_2, \dots, b_q\}$ a basis of W.

Back to solving $Ax=b$... but suppose I can't because it doesn't have a solution... Next best thing minimize $\|Ax-b\|$

closest point in the column space of A to the point b...

$$Ax = p = QQ^T b$$

since $p \in \operatorname{col} A$ then $Ax=p$ has a solution

what we just did...

Thus

$$QRx = QQ^T b$$

$$Q^T Q R x = Q^T Q Q^T b$$

$$Rx = Q^T b$$

the solution to this is the x that minimizes $\|Ax-b\|$

One more thing... orthogonal complement...

$$W^\perp = \{ y : y \cdot w = 0 \text{ for all } w \in W \}.$$

Theorem: 6.1

EM 3

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T$$

please read about this for next time...