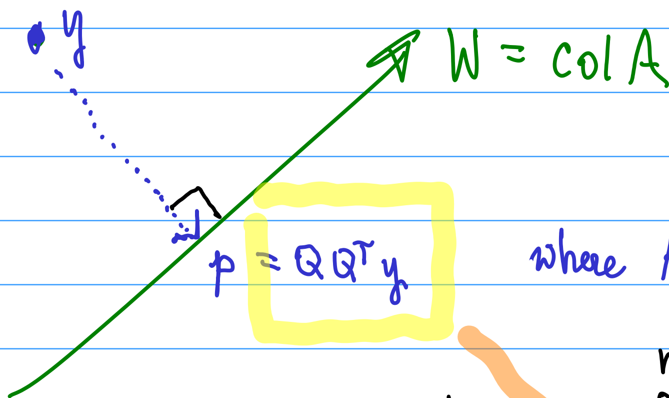


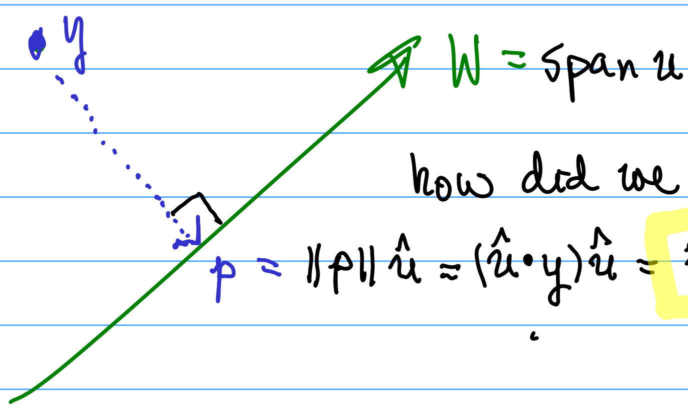
$p = \boxed{QQ^T y}$ where $A = QR$ ← upper triangular, square and invertible...

Q ← matrix with orthonormal columns

Since p is obtained by an orthogonal projection, then $p - y$ is perpendicular to any point $w \in W$.

In one dimension

Q is playing the role of $\hat{u}$ in higher dimensions



how did we find p in 1-d...

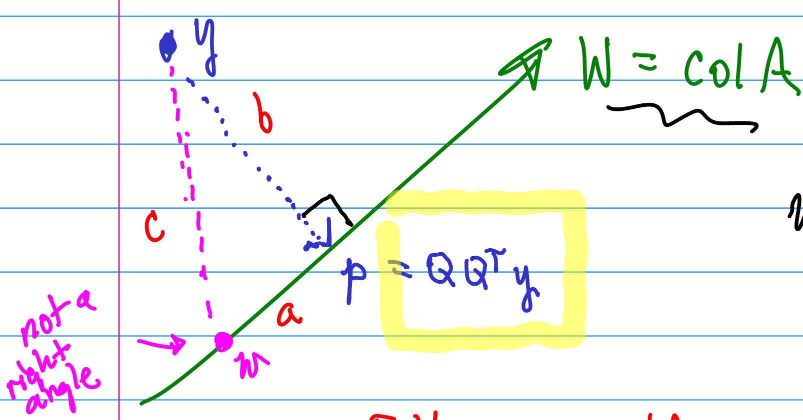
$p = \|p\| \hat{u} = (\hat{u} \cdot y)\hat{u} = \boxed{\hat{u}(\hat{u}^T y)}$

$\|p\| = \hat{u} \cdot y = \frac{u}{\|u\|} \cdot y = \cancel{\frac{(4,5) \cdot (5,2)}{\|(4,5)\|} = \frac{4\cdot 3 + 5 \cdot 2}{\sqrt{41}}} = \frac{22}{\sqrt{41}}$

Then what is p itself?

$p = \|p\| \hat{u} = \cancel{\frac{22}{\sqrt{41}} \frac{(4,5)}{\sqrt{41}} = \frac{22}{41}(4,5)}$

- The reason orthogonal project is useful is because p is the closest point in W to y



Why does the orthogonality condition give the minimum?

Pythagorean theorem: $a^2+b^2=c^2$

$$\|w-p\|^2 + \|p-y\|^2 = \|w-y\|^2$$

$$\|p-y\|^2 = \|w-y\|^2 - \|w-p\|^2 < \|w-y\|^2$$

so $\|p-y\| < \|w-y\|$ for all $w \in W$ such that $w \neq p$.

Therefore the closest point in W to y is given by QQ^Ty.

Solve

$Ax=b$ $\qquad A=QR$ $\qquad Q^TQ=I$

$QRx=b$

$\cancel{Q^TQ}Rx=Q^Tb$

$Rx=Q^Tb$

triangular and easy to invert

Last week

$$\underbrace{\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}}_{A \in \mathbb{R}^{4\times 3}} = \underbrace{\begin{bmatrix} \frac{3}{2\sqrt5} & \frac{1}{2\sqrt5} & \frac{-3}{2\sqrt5} \\ \frac{1}{2\sqrt5} & \frac{3}{2\sqrt5} & \frac{1}{2\sqrt5} \\ \frac{-1}{2\sqrt5} & \frac{3}{2\sqrt5} & \frac{1}{2\sqrt5} \\ \frac{3}{2\sqrt5} & \frac{-1}{2\sqrt5} & \frac{3}{2\sqrt5} \end{bmatrix}}_{Q \in \mathbb{R}^{4\times 3}} \underbrace{\begin{bmatrix} 2\sqrt5 & -4\sqrt5 & 3\sqrt5 \\ 0 & 2\sqrt5 & -\sqrt5 \\ 0 & 0 & 2\sqrt5 \end{bmatrix}}_{R \in \mathbb{R}^{3\times 3}}$$

What about $Ax = b$

& $\left\{ \begin{array}{l} 4 \text{ equations} \\ 3 \text{ unknowns} \end{array} \right\}$ overdetermined (only 3 pivots)

might not have a solution but

unless b is special this system will be inconsistent...

What does special mean? $b \in \operatorname{col} A$

If not, then what?

Idea: minimize the error

$E = \|Ax - b\|$ Find x that minimizes E.

Ax for all choices of x is $\operatorname{Col} A = W$

The closest point in W to b is just $p = QQ^T b$.

Thus $Ax = QQ^T b$ satisfies this minimization...

This does have a solution since $QQ^T b \in \operatorname{Col} A$.

$$QRx = QQ^T b$$
$$Q^T Q R x = Q^T Q Q^T b$$
$$Rx = Q^T b$$

what we had before...

6.5 Least Squares solution to $Ax=b$...

Idea: rewrite $Rx = Q^T b$ in terms of A ...

Claim this is in the book... → How does this equation look if its rewritten in terms of A ...

$A = QR$

$$A^TA = (QR)^T(QR) = R^T \cancel{Q^T Q} R = R^TR$$

product of invertible matrices is invertible

square, triangular and invertible

Therefore A^TA is an invertible matrix even though A itself wasn't even square...

Theorem: ~~If~~ the columns of A are linearly independent then A^TA is invertible.

Rewrite $Rx = Q^Tb$ in terms of A

$A = QR$ so $Q = AR^{-1}$ so $Q^T = (AR^{-1})^T = (R^{-1})^T A^T$

Therefore

$$Rx = (R^{-1})^T A^T b$$

$$x = R^{-1}(R^{-1})^T A^T b$$

Simplify

$$(R^{-1})(R^{-1})^T = R^{-1}(R^T)^{-1} = (R^TR)^{-1} = (A^TA)^{-1}$$

Answer $x = (A^TA)^{-1}A^Tb$

$$\boxed{A^TAx = A^Tb}$$

Normal equations... for solving $Ax=b$ by least squares...

EXAMPLE:

least-squares solution of $A\mathbf{x} = \mathbf{b}$.

15. $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}}_{R}$, $\mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

$Ax=b$ by least squares using QR.

$$Rx = \boxed{Q^Tb}$$

$$Q^Tb = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{14+6+1}{3} \\ \frac{-7+6-2}{3} \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$Rx = Q^Tb$

$$\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$3x_1 + 5x_2 = 7$

$x_2 = -1$

$x_1 = \dfrac{7 - 5(-1)}{3} = 4$

soln: $x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

Now work the same problem using the normal equations: $A^TAx = A^Tb$

$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 15 & 26 \end{bmatrix}$$

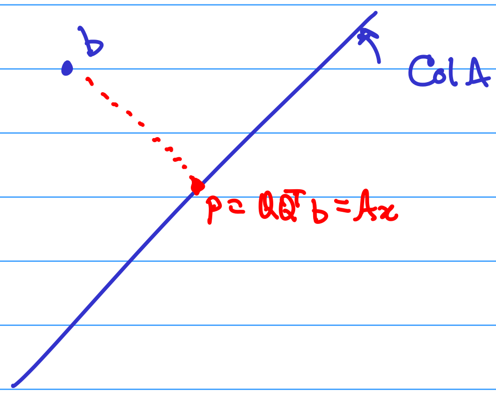
$$A^T b = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 34 \end{bmatrix}$$

Now solve

$$\begin{bmatrix} 9 & 15 \\ 15 & 26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 21 \\ 34 \end{bmatrix}$$

Check that $x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is the solution.

$$\begin{bmatrix} 9 & 15 \\ 15 & 26 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 36-15 \\ 60-26 \end{bmatrix} = \begin{bmatrix} 21 \\ 34 \end{bmatrix}$$



If $b \in \text{Col} A$ then $p = b$ so $\boxed{QQ^T b = b}$

could plug x back into $Ax = b$ to see if it worked...
and find $\|Ax - b\|$ to see how big it is...