The Singular Value Decomposition

Let A be an $m \times n$ matrix with rank r. Then there exists an $m \times n$ matrix Σ as in (3) for which the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and there exist an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$A = U\Sigma V^T$$

Gaussian elimination: row operation to obtain U from A

and the factorization

both L and U are triangular

Note, if row swaps are need when doing the elimination, they can be represented by a permutation matrix P.
Then
$A = PLU$

$$A = LU$$

echelon form of A

coefficients of the elimination steps

Note that P has orthonormal columns

$Ax = b$; $LUx = b$ solve the two systems $\begin{cases} Ly = b \\ Ux = y \end{cases}$

Gram-Schmidt algorithm: column operations

$Q^TQ = I$

triangular

$$A = QR$$

coefficients of the column operations

matrix with orthonormal columns that came from A

$Q^TQ = I$

$\|x\|^2 = x \cdot x$

$\|Qx\|^2 = Qx \cdot Qx$

$= Q^TQx \cdot x$

$= x \cdot x = \|x\|^2$

$Ax = b$; $QRx = b$; $Rx = Q^Tb$

If $Ax = b$ has a solution then solving $Rx = Q^Tb$ finds that solution..

projection of y onto Col A was given by QQ^Ty

If $Ax = b$ does not have a solution then $Rx = Q^Tb$ provides the best approximation in term of least squares.
I.e. minimize $\|Ax - b\|$.

Eigenvalue-eigenvector: Solved $Ax = \lambda x$ for λ and x.
we did this using determinants which only works for small matrices...

Sometimes we obtained n eigenvalues with n linearly independent eigenvectors.

$$S = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \qquad A = SDS^{-1}$$

eigenvectors

eigenvalues

diagonal

nothing special about S except how it appears in the factorization.

use of this was understanding powers of A since

$$A^\alpha = SD^\alpha S^{-1} = S \begin{bmatrix} \lambda_1^\alpha & & & 0 \\ & \lambda_2^\alpha & & \\ & & \ddots & \\ 0 & & & \lambda_n^\alpha \end{bmatrix} S^{-1}$$

$$e^{A} = S e^{D} S^{-1} = S \begin{bmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_d} \end{bmatrix} S^{-1}$$

Singular value decomposition $A = U\Sigma V^T$ — $V^T = V^{-1}$

U: orthogonal square and orthonormal columns, $U^T = U^{-1}$

Σ: diagonal

Geometric meaning of an orthogonal matrix... isometry... preserves lengths and angles...

To obtain the singular value decomposition we'll use the spectral theorem...

Spectral Theorem: If $A \in \mathbb{R}^{n \times n}$ and $A^T = A$ then

① the eigenvalues of A are real

$\boxed{2}$ the corresponding eigenvectors are perpendicular to each other

HW from chapter 5 that we worked in class

Then

$$S = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \cdots & \frac{x_n}{\|x_n\|} \end{bmatrix}$$

orthonormal columns and its square then $\underline{S^{-1} = S^T}$

$$A = SDS^{-1} = SDS^T$$

Now S is special not only because of where it appears in the factorization, but it's an orthogonal matrix..

The Spectral Theorem for Symmetric Matrices

An $n \times n$ symmetric matrix A has the following properties:

a. A has n real eigenvalues, counting multiplicities.

b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

d. A is orthogonally diagonalizable.

$A \in \mathbb{R}^{n\times n}$ and $A^T = A$

View this symmetry as a property about how A interacts with dot products..

Why are eigenvector that correspond to different eigenvalues perpendicular?

$$\lambda_1 \neq \lambda_2 \qquad Ax_1 = \lambda_1 x_1 \qquad Ax_2 = \lambda_2 x_2$$

Claim $\quad x_1 \cdot x_2 = 0.$

The transpose of a matrix came because that's what happens when a matrix jumps over a dot product..

$$Ax_1 \cdot x_2 = (Ax_1)^T x_2 = x_1^T A^T x_2 = x_1 \cdot A^T x_2 = x_1 \cdot Ax_2$$

because λ_2 is a scalar

$$\lambda_1 x_1 \cdot x_2 = x_1 \cdot \lambda_2 x_2$$

$$\lambda_1 x_1 \cdot x_2 = \lambda_2 x_1 \cdot x_2$$

The only way this equality could hold when $\lambda_1 \neq \lambda_2$ is for $x_1 \cdot x_2 = 0$

7.1 #20

$$A = \begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$$

Find eigenvalues and eigenvectors

note $A^T = A$

not a good pivot

$r_1 \leftrightarrow r_3$

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 8 & -4 \\ 8 & 5-\lambda & -4 \\ -4 & -4 & -1-\lambda \end{bmatrix} = -\det \begin{bmatrix} -4 & -4 & -1-\lambda \\ 8 & 5-\lambda & -4 \\ 5-\lambda & 8 & -4 \end{bmatrix}$$

$r_2 \leftarrow r_2 + 2r_1$

$r_3 \leftarrow r_3 + \frac{5-\lambda}{4} r_1$

$$= -\det \begin{bmatrix} -4 & -4 & -1-\lambda \\ 0 & -3-\lambda & -6-2\lambda \\ 0 & 3+\lambda & \underbrace{\frac{\lambda^2-4\lambda-21}{4}} \end{bmatrix}$$

$$-4 + \frac{(5-\lambda)(-1-\lambda)}{4} = \frac{-16-5-4\lambda+\lambda^2}{4}$$

$$= -(-4)\det \boxed{\begin{bmatrix} -3-\lambda & -6-2\lambda \\ 3+\lambda & \frac{\lambda^2-4\lambda-21}{4} \end{bmatrix}}$$

$$= 4(-3-\lambda)\det \begin{bmatrix} 1 & -6-2\lambda \\ -1 & \frac{\lambda^2-4\lambda-21}{4} \end{bmatrix}$$

$$= -(3+\lambda)\det \begin{bmatrix} 1 & -24-8\lambda \\ -1 & \lambda^2-4\lambda-21 \end{bmatrix}$$

$$= -(3+\lambda)\left[\lambda^2-4\lambda-21-24-8\lambda\right]$$

$$= -(3+\lambda)(\lambda^2-12\lambda-45) = -(3+\lambda)(\lambda-15)(\lambda+3) = 0$$

$3, 15$

eigenvalues $\lambda = -3$ with mult 2

$\lambda = 15$ with mult 1

eigenvector

$\lambda = 15$

$$A = \begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$$

one free var when done

$$\text{Nul}(A-\lambda I) = \text{Nul}\begin{bmatrix} -10 & 8 & -4 \\ 8 & -10 & -4 \\ -4 & -4 & -16 \end{bmatrix}$$

$r_1 \leftarrow \frac{1}{2} r_1$

$r_2 \leftarrow \frac{1}{2} r_2$

$r_3 \leftarrow \frac{-1}{4} r_3$

$r_1 \leftrightarrow r_3$

$$\text{Nul}\begin{bmatrix} -5 & 4 & -2 \\ 4 & -5 & -2 \\ 1 & 1 & 4 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 1 & 4 \\ 4 & -5 & -2 \\ -5 & 4 & -2 \end{bmatrix}$$

$r_2 \leftarrow r_2 - 4r_1$

since one free vble this last row has to be zero after elimination

$$= \text{Nul}\begin{bmatrix} 1 & 1 & 4 \\ 0 & -9 & -18 \\ 0 & 0 & 0 \end{bmatrix}$$

$r_2 \leftarrow \frac{1}{-9} r_2$

$$= \text{Nul}\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 - r_2$

$$= \text{Nul}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 2x_3 = 0$

$x_2 + 2x_3 = 0$

$x_1 = -2x_3$

$x_2 = -2x_3$

$x_3 = x_3$

$$x = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} x_3$$

eigenvector for $\lambda = 15$

dot products are zero

$\lambda = -3$

$$A = \begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$$

two free vbls

dot is zero

$$\text{Nul}(A - \lambda I) = \text{Nul}\begin{bmatrix} 8 & 8 & -4 \\ 8 & 8 & -4 \\ -4 & -4 & -4 \end{bmatrix} = \text{Nul}\begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$2x_1 + 2x_2 - x_3 = 0$

$x_1 = -x_2 + \frac{1}{2}x_3$

$$x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} x_3$$

- However these are not perpendicular to each other... we'll make them perpendicular using Gram-Schmidt next week...