13. Find the SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$ $\in \mathbb{R}^{2\times 3}$ [*Hint:* Work with A^T.]

Note C is the same as B after switching A with A^T.

$B = A^T A \in \mathbb{R}^{3\times 3}$ or $C = AA^T \in \mathbb{R}^{2\times 2}$

$3\times 2 \quad 2\times 3$ symmetric $\qquad 2\times 3 \quad 3\times 2$ symmetric

which to work with? B or C.

Find eigenvalues & eigenvectors of C

$$C = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

$17 \times 17 = 119 + 170 = 289$, $289 - 64 = 225$

$$\det(C - \lambda I) = \det \begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix} = (17-\lambda)^2 - 8^2 = \lambda^2 - 34\lambda + 289 - 64$$

$$= \lambda^2 - 34\lambda + 225 = (\lambda - 9)(\lambda - 25) = 0$$

$15 \times 15 = 75 + 150 = 225$

$225 = 15 \cdot 15 = 3 \cdot 5 \cdot 3 \cdot 5$

eigenvalues

$\lambda_1 = 25, \quad \lambda_2 = 9$

Find the eigenvectors of C.

$\lambda_1 = 25$

$$\text{Nul}(C - \lambda I) = \text{Nul}\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Since there is a free variable by the choice of λ we know this last row can be ignored.

Solve $x_1 - x_2 = 0$
$x_1 = x_2$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$

unit eigenvector for $\lambda = 25$ is $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 9$

$$\text{Nul}(C - \lambda I) = \text{Nul}\begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix} = \text{Nul}\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

solve $x_1 + x_2 = 0$
$x_1 = -x_2$

unit eigenvector for $\lambda = 9$ is $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

At this point I know two matrices in the singular value decomposition: Σ and U.

by working with A^T I get the factorization of A^T

$$A^T = V\Sigma U^T$$

matrix with orthonormal columns that is square...

orthogonal matrices which means $V^T = V^{-1}$ and $U^T = U^{-1}$

$$\underbrace{\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}}_{3\times 2} = \underbrace{V}_{3\times 3} \underbrace{\Sigma}_{3\times 2} \underbrace{U^T}_{2\times 2}$$

diagonal

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

what's V? Since we're working with A^T we take the eigenvectors in U and apply A^T to them.

Check this is really a unit vector... it is...

$$y_1 = A^T x_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} \qquad z_1 = \frac{y_1}{\|y_1\|} = \frac{y_1}{\sqrt{\lambda_1}} = \frac{1}{5}\frac{1}{\sqrt{2}}\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$y_2 = A^T x_2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} \qquad z_2 = \frac{y_2}{\sqrt{\lambda_2}} = \frac{1}{3}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} = \frac{1}{\sqrt{18}}\begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{18} \\ 1/\sqrt{18} \\ -4/\sqrt{18} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & \\ 1/\sqrt{2} & 1/\sqrt{18} & ? \\ 0 & -4/\sqrt{18} & \end{bmatrix}$$

This is actually the eigenvector corresponding to the zero eigenvalue of the B matrix.

make up this missing vector so that V is square with orthonormal columns...

I want a unit vector that's perpendicular to the other vectors...

This vector is always perpendicular to the first

$$\begin{bmatrix} -a \\ a \\ b \end{bmatrix}$$

need to choose a and b so it's also perpendicular to the second column:

Thus...

$$\begin{bmatrix} -a \\ a \\ b \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} = a + a - 4b = 2a - 4b = 0$$

$$a = 2b$$

$$\begin{bmatrix} -2b \\ 2b \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} b \quad \text{unit vector is } \frac{1}{\sqrt{9}} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

the missing vector

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & -4/\sqrt{18} & 1/3 \end{bmatrix}$$

Thus $A^T = V \Sigma U^T$ is

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & -4/\sqrt{18} & 1/3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

A^T = orthogonal matrix, diagonal matrix, orthogonal matrix

What is the factorization of A?

transpose

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A = A^{TT} = (V \Sigma U^T)^T = U^{TT} \Sigma^T V^T = U \Sigma^T V^T$$

This is the SVD of the matrix A:

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{18} & 1/\sqrt{18} & -4/\sqrt{18} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

A = orthogonal, diagonal, orthogonal

(i) Whenever a system has free variables, the solution set contains a unique solution.

(A) True

(B) False

The free variables give the flexibility for lots of solutions...

(ii) An inconsistent system has more than one solution.

(A) True

(B) False

actually, inconsistent means no solutions.

(iii) When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

(A) True

(B) False

The product of two matrices is defined by the composition of the linear transformations.

allows us to define matrix multiplication

(iv) $\det(A^{-1}) = 1/\det(A)$.

(A) True

(B) False

$\det(I)=1$ by definition

$\det(A^{-1}A)=1$ so $(\det A^{-1})(\det A)=1$

(v) Cramer's rule can only be used for invertible matrices.

(A) True

(B) False

$x_i = \dfrac{\det A_i(b)}{\det A}$ then $Ax=b$

if A is not invertible either its not square so $\det A$ makes no sense or $\det A=0$

(vi) If W is a subspace of $\mathbf{R}^n$ and v is in both W and $W^{\perp}$, then $v = 0$.

(A) True

(B) False

$v \in W$ and $v \in W^{\perp}$ means $v \cdot v = 0$

the only vector like this is $v=0$

(vii) If $A = QR$ where Q has orthonormal columns, then $R = Q^T A$.

(A) True