Chapter 1.7

means v_1 has n elements

no free variables

$Ax=0$ has only one solution

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

linear combination

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$Ax=0$ has many solutions

There are free variables

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \quad (2)$$

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix} \in \mathbb{R}^{n \times p}$$

The same

$$\underset{n\times p}{A}\,\underset{p}{x} = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = v_1x_1 + v_2x_2 + \cdots + v_px_p$$

So all I'm doing is considering the equation $\longrightarrow$ $Ax=0$

Note from 1.5 that $Ax=0$ is never inconsistent because $x=0$ is always a solution

Therefore...

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Let $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$ and

$Ax=0$ has lots of solutions... only one of which is zero...

$\bar{\bar{A}}\vec{x} = \vec{0}$ and $x \neq 0$

$v_1x_1 + v_2x_2 + \dots + v_px_p = 0$

$$\boxed{p=4: \quad (v_1)(0) + (v_2)(2) + (v_3)(3) + (v_4)(0) = 0}$$

$x = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}$

then ... $v_3 = -\frac{2}{3}v_2$

$\overbrace{\text{more vectors}}^{p}$ $>$ $\overbrace{\text{entries in each vector}}^{n}$

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\underbrace{\{\mathbf{v}_1, \dots, \mathbf{v}_p\}}_{p \text{ vectors}}$ in $\mathbb{R}^{n}$ (size of each vector) is linearly dependent if $p > n$.

rows cols

$$A = \begin{bmatrix} \overset{1}{v_1} & \overset{2}{v_2} & \dots & \overset{p}{v_p} \end{bmatrix} \in \mathbb{R}^{n \times p}$$

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix}$

$$A = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 6 \\ 2 & 3 & 1 & -1 \\ 3 & -2 & 1 & 3 \end{bmatrix} \in \mathbb{R}^{3 \times 4}$$

is $4 > 3$ $\boxed{\text{yes!}}$

If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

I need $Ax = 0$ to have infinite # of solutions

$$Ax = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 3 & -2 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix}$$

So any vector $x = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$ gives that $Ax = 0$ in this case.

Read 1.8 ... and we'll start 1.9