Tuesday Feb 22
Section 1005

Finish chapter 2.2

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\underbrace{ad - bc}_{\text{This is a determinant ... Next chapter}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

Formula to find inverse of a 2x2 matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \qquad A^{-1} = \frac{1}{\underbrace{7-6}_{=1}} \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Check:

so $a=1$ $b=3$
$c=2$ $d=7$

$$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Identity   identity

Find the inverse using augmented matrix

row operations

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

The Reduced-Row Echelon form is always an identity matrix?

Yes... The reduced row eschelon form of an invertible matrix is always the identity matrix...

$r_2 \leftarrow r_2 - 2r_1$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$r_1 \leftarrow r_1 - 3r_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ or $\begin{cases} x_1 + 3x_2 = -1 \\ 2x_1 + 7x_2 = 6 \end{cases}$

$Ax = b$ solve using the inverse matrix...

$$A^{-1}Ax = A^{-1}b$$

$$x = A^{-1}b = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} -25 \\ 8 \end{bmatrix}$$

Check by plugging the answer back in

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -25 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$, $\quad A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad (A^{-1})^T = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

This is the same as $(A^T)^{-1}$

$(A^{-1})^T = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$ $\qquad A^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (A^{-1})^T = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

? Identity?

$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Identity $\qquad$ ← identity

The definition of the inverse

$$AA^{-1} = I$$

About transposes: $\boxed{(AB)^T = B^T A^T}$

$$(AA^{-1})^T = \boxed{(A^{-1})^T A^T} = I^T = \boxed{I}$$

Thus; $(A^{-1})^T A^T = I$

The definition of inverse means $(A^{-1})^T$ is the inverse of A^T.

$(A^T)^{-1}$ is

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

difficult

maybe not so difficult

What does it mean again that the colums form a linearly independent set.

recall definition of linearly independent from chapter 1.7

means v_1 has n elements

no free variables

$Ax=0$ has only one solution

vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent**
ion

linear combination

$$\boxed{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}}$$

l solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be linearly dependent

$$A = \begin{bmatrix} v_1 | v_2 | \cdots | v_p \end{bmatrix} \quad \text{same equation} \quad \boxed{Ax = 0}$$

Moving on to chapter 2.4

Please read

EXAMPLE 3 Let

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

Chapter 2.5

LU factorization problem:

don't need to know A explicitly to solve $Ax=b$

6. $A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ -3 & -6 & -7 & 2 \\ 3 & 3 & 0 & -4 \\ -5 & -3 & 2 & 9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

How $LUx = b$

system $\begin{cases} Ly = b \\ Ux = y \end{cases}$

First solve $Ly = b$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$$

by substitution ...

$$\begin{cases} y_1 = 1 \\ -3y_1 + \boxed{y_2} = -2 \\ 3y_1 - 2y_2 + \boxed{y_3} = -1 \\ -5y_1 + 4y_2 - y_3 + y_4 = 2 \end{cases}$$

$y_1 = 1$

$y_2 = -2 + 3y_1 = -2 + 3 \cdot 1 = 1$

$y_3 = -1 - 3y_1 + 2y_2 = -1 - 3 + 2 = -2$

$y_4 = 2 + 5y_1 - 4y_2 + y_3$

$= 2 + 5 - 4 - 2 = 1$

Thus $y = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

Now solve $Ux = y$

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

by back-substitution

$$\begin{cases} \boxed{x_1} + 3x_2 + 4x_3 = 1 \\ 3\boxed{x_2} + 5x_3 + 2x_4 = 1 \\ -2x_3 = -2 \\ x_4 = 1 \end{cases}$$

start with the last pivot variable

$x_4 = 1$

$x_3 = \frac{-2}{-2} = 1$

$x_2 = \frac{1 - 5x_3 - 2x_4}{3}$

$= \frac{1 - 5 - 2}{3} = \frac{-6}{3} = -2$

$x_1 = 1 - 3x_2 - 4x_3 = 1 - 3(-2) - 4(1)$

$= 1 + 6 - 4 = 3$

$x = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}$ Answer