A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Examples of column space and Nullspace

25. $A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix} \in \mathbb{R}^{4\times 5}$

$r_2 \leftarrow r_2 + r_1$
$r_3 \leftarrow r_3 + 2r_1$
$r_4 \leftarrow r_4 - 3r_1$

$\sim \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← not the reduced row echelon form of A

$r_1 \leftarrow r_1 - 2r_2$

$\begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$r_2 \leftarrow \frac{1}{2} r_2$

$R = \begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 1 & 5/2 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← reduced row echelon form of A

range of the linear function
$\text{Col}(A) = \{ A x : x \in \mathbb{R}^5 \}$

$\text{Nul}(A) = \{ x : A x = 0 \}$
solutions to the homogeneous eq.

(a) Is $0 \in \text{Col}(A)$?
Is $0 \in \text{Nul}(A)$?

Since $A0 = 0$ then $0 \in \text{Col}(A)$.
$\not\Rightarrow$ then $0 \in \text{Nul}(A)$

(b) Suppose $u \in \text{Col}(A)$, $v \in \text{Col}(A)$
why is $u+v \in \text{Col}(A)$?

Since $u \in \text{Col}(A)$ then $u = Ax$ for some x
$v \in \text{Col}(A)$ then $v = Ay$ for some y

$u+v = Ax + Ay = A(\underbrace{x+y})$
input that makes the output $u+v$

Thus $u+v \in \text{Col}(A)$.

actually compute the nullspace

Solve $Ax=0$ ... find all solutions... that's Nul(A) by definition...

$$Ax = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Same solutions as

$$
\begin{array}{c}
\begin{array}{ccccc} P & P & F & P & F \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
R x = \begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 1 & 5/2 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{array}
$$

or

$$
\left\{ \begin{array}{l} x_1 - 2x_3 + 7x_5 = 0 \\ x_2 + \frac{5}{2} x_3 - \frac{1}{2} x_5 = 0 \\ x_4 + 4x_5 = 0 \end{array} \right.
$$

$$x_4 = -4x_5$$
$$x_2 = -\tfrac{5}{3} x_3 + \tfrac{1}{2} x_5$$
$$x_1 = 2x_3 - 7x_5$$

Vector form of soln.
factor out free vbls

Solution

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 7x_5 \\ -\frac{5}{3}x_3 + \frac{1}{2}x_5 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ -5/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -7 \\ 1/2 \\ 0 \\ -4 \\ 1 \end{bmatrix} x_5
$$

$$\mathrm{Nul}(A) = \{ x : Ax = 0 \} = \mathrm{Nul}(R) = \{ x : Rx = 0 \}$$

inputs to A that give zero...

$$
= \left\{ \begin{bmatrix} 2 \\ -5/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -7 \\ 1/2 \\ 0 \\ -4 \\ 1 \end{bmatrix} x_5 : x_3, x_5 \in \mathbb{R} \right\}
$$

outputs of a different matrix

$$
= \left\{ \begin{bmatrix} 2 & -7 \\ -5/3 & 1/2 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} : x_3, x_5 \in \mathbb{R} \right\} = \mathrm{Col}(N) \text{ where } N = \begin{bmatrix} 2 & -7 \\ -5/3 & 1/2 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{bmatrix}
$$

Two linearly ind. columns that span the nullspace.

These 1's correspond to the free variable equal to itself

$$\mathrm{Col}(N) = \{ Nw : w \in \mathbb{R}^2 \}$$

Does Nul(A) satisfy property b.

b. For each **u** and **v** in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

Let $u \in \mathrm{Nul}(A)$ and $v \in \mathrm{Nul}(A)$ then $Au = 0$ and $Av = 0$

Then $A(u+v) = Au + Av = 0 + 0 = 0$ shows $u + v \in \mathrm{Nul}(A)$

Note any subspace can be written as $\mathrm{Col}(A)$ for some matrix A.

$$\mathrm{Col}\left(\begin{array}{ccccc} P & P & F & P & F \\ \left[\begin{matrix} 1 \\ -1 \\ -2 \\ 3 \end{matrix}\right. & \begin{matrix} 4 \\ 2 \\ 2 \\ 6 \end{matrix} & \begin{matrix} 8 \\ 7 \\ 9 \\ 9 \end{matrix} & \begin{matrix} -3 \\ 3 \\ 5 \\ -5 \end{matrix} & \left.\begin{matrix} -7 \\ 4 \\ 5 \\ -2 \end{matrix}\right] \end{array}\right) = \left\{ \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\}$$

eliminate dependencies in span

$$= \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix} x_2 + \begin{bmatrix} 8 \\ 7 \\ 9 \\ 9 \end{bmatrix} x_3 + \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix} x_4 + \begin{bmatrix} -7 \\ 4 \\ 5 \\ -2 \end{bmatrix} x_5 : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\}$$

Goal simplify this... using the reduced row echelon form of A:

$$R = \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ P & P & F & P & F \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} -2 \\ 5/2 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} 7 \\ -1/2 \\ 4 \\ 0 \end{matrix}\right] \end{array}$$

easily see dependency relation in R.

These vectors are independent

$$\overset{x_3}{\begin{bmatrix} -2 \\ 5/2 \\ 0 \\ 0 \end{bmatrix}} = \overset{x_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}(-2) + \overset{x_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}(5/2) \qquad \overset{x_5}{\begin{bmatrix} 7 \\ -1/2 \\ 4 \\ 0 \end{bmatrix}} = \overset{x_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}(7) + \overset{x_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}\left(-\tfrac{1}{2}\right) + \overset{x_4}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}(4)$$

so these are too

$$\begin{bmatrix} 8 \\ 7 \\ 9 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}(-2) + \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}(5/2) \qquad \begin{bmatrix} -7 \\ 4 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}(7) + \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}\left(-\tfrac{1}{2}\right) + \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}(4)$$

changes made in green were typos in the original

$$\mathrm{Col}(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}(x_1 - 2x_3 + 7x_5) + \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}\left(x_2 + \tfrac{5}{2}x_3 - \tfrac{1}{2}x_5\right) + \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}(x_4 + 4x_5) : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\}$$

Thus...

$$\underset{\text{big original matrix}}{\mathrm{Col}(A)} = \left\{ \underset{\text{smaller matrix}}{\begin{bmatrix} 1 & 4 & -3 \\ -1 & 2 & 3 \\ -2 & 2 & 5 \\ 3 & 6 & -5 \end{bmatrix}} w : w \in \mathbb{R}^3 \right\}$$

in summary

$$\mathrm{Col}(A) = \mathrm{Col}\left(\begin{bmatrix} 1 & 4 & -3 \\ -1 & 2 & 3 \\ -2 & 2 & 5 \\ 3 & 6 & -5 \end{bmatrix}\right)$$

which is just the columns of A that correspond to the pivots in R

independent columns that span Col(A)

Definitions

Independent vectors that span a subspace are called a $\boxed{\text{basis}}$ for that subspace...

$\boxed{\text{Dimension}}$ of the subspace is the number of vectors in a basis

$\dim \mathrm{Col}(A) = 3 =$ # of pivot variables..

$\dim \mathrm{Nul}(A) = \dim \mathrm{Col}(N) = 2 =$ # of free variables...

Theorem $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) =$ # of columns of $A = 5$