$A \in \mathbb{R}^{m \times n}$

The Rank Theorem

If a matrix A has n columns, then rank A + dim Nul $A = n$.

{# of columns in A

rank A = dim Col(A)

# of pivot vbls + # of free vbls = # of columns

The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

What's a basis of H? A set of vectors $\{w_1, w_2, \dots, w_p\}$ such that

- The w_i's are independent
- The w_i's span the subspace H

Strategies for finding a basis

1. Start with too many that span H and throw away dependent vectors until what's left is independent..
2. Start with a few independent vectors and add more independent vectors until you get something that spans.

② usually works better... related to Gram-Schmidt process!..

① We used this method to find a basis for Col(A)

There could be a problem if one starts with an infinite spanning set.

- a set of vectors $\{w_1, \dots, w_p\}$ $\boxed{\text{span}}$ the space H means for every $b \in H$ the equation $Ax = b$ has at least one solution where

$$A = \begin{bmatrix} w_1 & w_2 & \dots & w_p \end{bmatrix}$$

not necessarily square...

- a set of vectors $\{w_1, \dots, w_p\}$ is independent means $Ax = 0$ has only the solution $x = 0$.

This happen when there are no free variables...

The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

Since H is p-dimensional it has a basis with p vectors.

Let $\{w_1, w_2, \dots, w_p\}$ be a basis for H.

Suppose $\{v_1, v_2, \dots, v_p\}$ is a set of vectors that span H.

$$A = \underbrace{\begin{bmatrix} w_1 & w_2 & \dots & w_p \end{bmatrix}}_{\text{lin. ind. and span}} \in \mathbb{R}^{m \times p} \qquad B = \underbrace{\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}}_{\text{these span}} \in \mathbb{R}^{m \times p}$$

Since the vectors v_i span H and $w_i \in H$ then

$$\left.\begin{aligned} w_1 &= v_1 c_{1,1} + v_2 c_{21} + \dots + v_p c_{p,1} \\ w_2 &= v_1 c_{1,2} + v_2 c_{22} + \dots + v_p c_{p,2} \\ &\vdots \\ w_p &= v_1 c_{1,p} + v_2 c_{2,p} + \dots + v_p c_{p,p} \end{aligned}\right\} \text{ for some } c_i\text{'s}.$$

Let

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \ddots & c_{2p} \\ \vdots & & & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pp} \end{bmatrix} \in \mathbb{R}^{p \times p}$$

check the order of the indices

Therefore $A = BC$

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \ddots & c_{2p} \\ \vdots & & & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pp} \end{bmatrix}$$

$$\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix} \begin{bmatrix} & & & \end{bmatrix}$$

$A = BC$

Since the columns of A are linearly independent $Ax=0$ has only the solution $x=0$.

Suppose $Cx=0$ had lots of solutions...

Then $BCx = B0 = 0$ also has lots of solutions

But then $BC=A$ means $Ax=0$ would have lots of solutions. Therefore $Cx=0$ could only have one solution.

invertible matrix theorem

- C has no free variables
- C has a pivot in each column.
- C is square
- C has a pivot in each row
- C is invertible

Trying to see why the columns of B are independent..

$$A = BC$$

Thus $AC^{-1} = B\cancel{CC^{-1}}$

implies $B = AC^{-1}$

Need to show $Bz=0$ has only the solution $z=0$.

Solve $AC^{-1}z = 0$. Equivalently solve this system of systems...

$$\begin{cases} Ay = 0 \\ C^{-1}z = y \end{cases}$$

only solution is $y = 0$ since columns of A are linearly independent

$C^{-1}z = 0$

$z = C0 = 0$

So $z = 0$

Therefore, the columns of B are linearly independent.