Definition

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

formula

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

* use this formula and all the properties below are satisfied...

Notation

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Theorem 1

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Theorem 3

det is a function that maps square matrices into numbers

Row Operations

Let A be a square matrix.

Elimination step

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$. $r_i \leftarrow r_i - \alpha r_j$

row swap

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$. $r_i \leftrightarrow r_j$, $i \neq j$

scaling operation

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

$r_i \leftarrow k r_i$ $k \neq 0$

$\det[3] = 3\det[1] = 3$

one more property

Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

$A \in \mathbb{R}^{n \times n}$ $\quad$ $I \in \mathbb{R}^{n \times n}$

$AI = A$ $\quad$ and $\quad$ $II = I$

$\det A = \det(A)\det(I)$ $\qquad$ $\det(I) = \det(I)\det(I)$

same as the equation

$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

either $x = 0$ or $x = 1$

thus either

$\det I = 0$ or $\det I = 1$.

$\det(A) = \det(A)\, x$

$\det(A)x - \det A = 0$

$\det(A)(x-1) = 0$

either $\det A = 0$ or $\det I = 1$

which one? If $\det I = 0$ then all invertible matrices would also have a zero determinant. That's not very useful...

- The only other choice (the sensible one) is for $\det I = 1$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\underset{n\times n}{\det A} = a_{11} \det \underset{n-1\times n-1}{A_{11}} - a_{12} \det \underset{n-1\times n-1}{A_{12}} + \cdots + (-1)^{1+n} a_{1n} \det \underset{n-1\times n-1}{A_{1n}}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

$(n-1)\times(n-1)$

What does A_{ij} mean?

Crazy meaning: Create a new matrix for A by crossing out the ith-row and jth column. This new matrix is called A_{ij}.

Suppose $A \in \mathbb{R}^{10 \times 10}$ how many terms are there in the definition of $\det A$?

$10 \cdot 9 \cdot 8 \cdots 2 \cdot 1 = 10! = n!$

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julia> factorial(10)
3628800

julia> 1*2*3*4*5*6*7*8*9*10
3628800
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How many terms for Gaussian elimination?

10 pivots...

9 elimination operations for each pivot.

$r_i \leftarrow r_i - \alpha r_j$

involves 10 terms...

1 scaling operation for each pivot

$r_i \leftarrow k r_i$ involves 10 terms...

Total number of terms $10^3 = n^3$.

$= 1000$

break even point is about $n = 5$.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix}$$

Use the definition to find det A ...

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \underbrace{A_{1j}}_{(n-1)\times(n-1)}$$

$$\det A_{11} = \det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix} = 5 \cdot 1 - 3 \cdot 1 = 2$$

$$\det \begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix} = \det [1] = 1$$

$$\det \begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix} = \det [1] = 1$$

a_{12}

$$\det A_{12} = \det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix} = \det \underbrace{\begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix}}_{A_{12}} = 2 \cdot 1 - 3 \cdot (-3) = 11$$

$$\det A_{13} = \det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} = 2 \cdot 1 - 5(-3) = 17$$

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$= 1 \cdot 2 - 2 \cdot 11 + (-1) \cdot 17 = -37$$

$$\begin{array}{r} 22 \\ 17 \\ \hline 39 \end{array}$$

Therefore ...

$$\det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix} = -37$$

Theorem 2: If the matrix is triangular then computing the determinant is super easy...

$$\det A = \det\begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ -3 & 1 & 1 \end{bmatrix} = a_{11}\det A_{11} - \overset{0}{a_{12}}\det A_{12} + \overset{0}{a_{13}}\det A_{13}$$

$$= 1 \cdot 5 \cdot 1$$

product along the diagonal.

$$\det A_{11} = \det\begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\det A_{12} = \det\begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\det A_{13} = \det\begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\det A_{11} = \det\begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} = 5 \cdot 1 - 0 \cdot ?$$

Another example...

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 8 \\ 0 & 0 & 2 \end{bmatrix} = 1 \cdot 6 \cdot 2 = 12.$$