Definition of determinant:

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$ then $\det A = a_{11}a_{22} - a_{12}a_{21}$

two terms...

If $A \in \mathbb{R}^{n\times n}$ then

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n}a_{1n}\det A_{1n}$$
$$= \boxed{\sum_{j=1}^{n}(-1)^{1+j}a_{1j}}\det A_{1j}$$

Note! always cross out the first row

pattern for extending an $n-1 \times n-1$ determinant to a $n \times n$ deterinant..

Example

$A = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 5 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

Need to know What A_{ij} means for the recursive definition to make sense

A_{pq} means create a smaller matrix by crossing out the pth row and qth column.

$a_{11} = 1$

$A_{11} = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 5 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 2 & 1 \end{bmatrix}$

what's left

$\det A_{11} = 5\cdot 1 - 2\cdot 4$
$= -3$

$a_{12} = 2$

$$A_{12} = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 5 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix}$$

$\det A_{12} = 3 \cdot 1 - 4(-1)$
$= 7$

$a_{13} = -2$

$$A_{13} = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 5 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$$

$\det A_{13} = 3 \cdot 2 - 5(-1)$
$= 11$

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

a function that map matrices into numbers..

$22 + 14 = 36$, $36 + 3 = 39$

Thus,

$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$

elements of the matrix A

minors of the matrix A

$= (1)(-3) - (2)(7) + (-2)(11)$

$= -3 - 14 - 22 = -39$

Note to compute the determinant of a 3x3 matrix I need to find the determinants of 3 2x2 matrices and combine them together.

Note to compute the determinant of a 4x4 matrix I need to find the determinants of 4 3x3 matrices and combine them together.

24 terms total.

4 in total it takes $4 \cdot 3 = 12$ 2×2 matrix determinants to find the determinant of a 4×4 matrix.

Note to compute the determinant of a 5×5 matrix I need to find the determinants of 5 4×4 matrices and combine them together.

120 terms total

in total it takes $5\cdot 4\cdot 3 = 60$ 2×2 matrix determinants to find the determinant of a 5×5 matrix.

The number of terms to compute the determinant of an $n\times n$ matrix is $n!$

too many terms even when n is relatively small for reasonable computations.

Alternative way to compute the determinant is based on Gaussian elimination. That is a bunch of row operation...

Row Operations

determinants were designed to interact with row operations in a nice way...

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$. $r_i \leftarrow r_i - \alpha r_j$ $i \neq j$

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$. $r_i \leftrightarrow r_j$ $i \neq j$

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

$r_i \leftarrow k r_i$ where $k \neq 0$.

Why?

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

cross out the first row and scan across the columns

this $(-1)^{1+j}$ is distinctive...

cross out ith row and scan across the columns

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

where $C_{ij} = (-1)^{i+j} \det A_{ij}$ ← cofactors

ansion down the jth column is

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

cross out the jth columns and scan across the rows

Note since you can swap columns with rows and get the same answer when computing, then

$\det A = \det A^T$.

REM 2

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Example

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$ $a_{11} = 0$

what is $\det A = 1 \cdot 3 \cdot 6 \cdot 10 = 180$

only one term

$$\det A = a_{11} \det A_{11} - \underbrace{a_{12}}_{=0} \det A_{12} + \underbrace{a_{13}}_{=0} \det A_{13} - \underbrace{a_{14}}_{=0} \det A_{14}$$

$$= 1 \det \begin{bmatrix} 3 & 0 & 0 \\ 5 & 6 & 0 \\ 8 & 9 & 10 \end{bmatrix} = 1 \cdot 3 \cdot \det \begin{bmatrix} 6 & 0 \\ 9 & 10 \end{bmatrix} = 1 \cdot 3 \cdot 6 \cdot 10$$

$$\det \begin{bmatrix} 2 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix} = -180$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

$r_2 \leftarrow r_2 - 3r_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \qquad \det \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = 1 \cdot (-2) = -2$$

The properties of determinant involving row operations combined with

$\det I = 1$

is enough to figure out that formula. Is it?

$$\det I = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

If you use row operations to compute a determinant it take about n^3 operations.