Nonlinear Equations

1a. Consider the equation
$$x + \sin x = c. \tag{1}$$
Use the intermediate value theorem to show that this equation has a solution for every real number c.

b. Let $h(x) = x + \sin x$. Show that $h'(x) > 0$ except at the isolated points $(2n + 1)\pi$ where n is an integer. Conclude that $h : \mathbf{R} \to \mathbf{R}$ is strictly increasing, and therefore, that equation (1) has exactly one solution x corresponding to each real number c.

c. Find the approximate solution x^* of equation (1) to machine precision for $c = 3$ using the bisection method.

d. Let $\Phi(x) = c - \sin x$. Then $\Phi(x) = x$ if and only if x is a solution to equation (1). Let $c = 3$ and recursively define $x_{n+1} = \Phi(x_n)$ where $x_0 = c$. Does x_n converge to x^*?

e. Define the errors $e_n = x_n - x^*$ where x^* is the best approximation found in part c and x_n are the approximations found in part d. If e_{n+1} is related to e_n by
$$|e_{n+1}| \approx K|e_n|^p$$
then the points $(\log|e_n|, \log|e_{n+1}|)$ should lie on a straight line with slope p. Note that p is called the order of convergence. Plot the points and find the order of convergence for the sequence x_n from part d.

f. Use Newton's method
$$\Phi(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x + \sin x - c}{1 + \cos x}$$
with $c = 3$ to compute another sequence x_n converging to x^*. Find the order of convergence for Newton's method.

g. [Extra Credit and for CS/Math 666] Use the bisection method to compute the solution x to equation (1) when $c = \pi$ starting with an initial bracket of $[3, 4]$. Repeat starting with the initial bracket $[3, 3.2]$. Try the bracket $[3.1, 4]$. Compute your approximations as accurately as possible. Are your answers the same given the different starting brackets? If not, are any correct? If not, what is the correct answer? Explain.