Gaussian Quadrature

1a. Consider the Gaussian quadrature rule for $\int_{-1}^{1} f(x)dx$ given by

$$G(n,f)=\sum_{i=0}^{n} w_i f(x_i)$$

where x_i are the roots of the Legendre polynomial of degree $n+1$ and w_i are given by the Vandermond system $V^T w = y$ where

$$V=\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \quad \text{and} \quad y=\int_{-1}^{1}\begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix} dx.$$

Write program to calculate $G(n,f)$ for $n=1,\ldots,10$. Hint: you may use the Maple script `legendre.mpl` to precompute the values of x_i and w_i needed for your quadrature routine.

1b. Let $f(x)=1/(1+x^2)$. Compute the integral $\int_{-1}^{1} f(x)dx$ exactly and then compute $G(n,f)$ for $n=1,\ldots,10$.

1c. Using a change of variables show that

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\Big(\frac{t(b-a)+a+b}{2}\Big)dt.$$

Gaussian quadrature on the interval $[a,b]$ is thus given by

$$G(n,f,a,b)=\frac{b-a}{2}G(n,g) \quad \text{where} \quad g(t)=f\Big(\frac{t(b-a)+a+b}{2}\Big).$$

Write a program to compute $G(n,f,a,b)$ for $n=1,\ldots,10$.

1d. Let $h=b-a$ and define

$$E_h = G(n,f,a,b)-\int_a^b f(x)dx.$$

Theorem 5.5.1 implies $E_h \approx ch^{2n+3}$. It follows, upon considering this same estimate for the intervals $[a, m]$ and $[m, b]$ where $m = (a + b)/2$, that

$$E_h \approx \frac{G(n, f, a, b) - G(n, f, a, m) - G(n, f, m, b)}{1 - 2^{-2n-2}}.$$

Compute E_h exactly for the approximations found in part 1b and compare these to the approximations for E_h given above.

1e. [Extra Credit and for CS/Math 666] Use the estimates in part 1d to create a recursive integrating routine with tolerance ϵ given by

$$G_{\mathrm{r}}(n, f, a, b, \epsilon) = \begin{cases} G(n, f, a, b) & \text{if } E_h \le \epsilon \\ \\ G_{\mathrm{r}}(n, f, a, (a+b)/2, \epsilon/2) & \\ \quad + G_{\mathrm{r}}(n, f, (a+b)/2, b, \epsilon/2) & \text{otherwise.} \end{cases}$$

Compute $G_{\mathrm{r}}(3, f, -1, 1, 10^{-12})$.