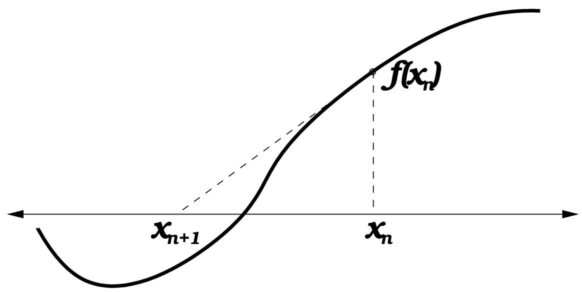
Newton's Method

Newton's method may be viewed as a way of creating a function Φ such that the fixed point iteration $x_{n+1} = \Phi(x_n)$ converges rapidly to a solution of $f(x) = 0$ for all initial value of x_0 sufficiently near the solution. You've probably met Newton's method in Calculus class. There it was derived by setting x_{n+1} equal to the root of the tangent line approximation to f through the point $(x_n, f(x_n))$. Graphically we have

Newton's Method



Therefore, equating the slope of the tangent line to the derivative at x_n, we obtain

$$\frac{f(x_n) - 0}{x_n - x_{n+1}} = f'(x_n).$$

Solving for x_{n+1} yields

$$x_{n+1} = \Phi(x_n) \qquad \text{where} \qquad \Phi(x) = x - \frac{f(x)}{f'(x)}.$$

Newton's method was proposed by Issac Newton in 1669 as a way to find the roots of polynomial equations and was used by Heron the elder in 100BC to approximate $\sqrt{a}$. All of this happened before digital computers became widespread. Newton's method is such a good method for solving non-linear equations, that we still use it today.

Recall that for Φ to be suitable for solving $f(x) = 0$ by fixed point iteration it should satisfy the conditions

1. $\Phi(x) = x$ if and only if $f(x) = 0$, and
2. $|\Phi'(x)| < 1$ in a neighborhood of the solution.

Let us check these conditions for the Φ given by Newton's method. First note, provided $f'(x) \neq 0$, that $f(x) = 0$ if and only if $\Phi(x) = x$. Now we assume f is two times continuously differentiable and differentiate to obtain

$$\Phi'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

Let $x = \alpha$ be a solution to $f(x) = 0$. Then $\Phi'(\alpha) = 0$. It follows, by continuity, that $\Phi'(x)$ is close to zero when x is close to α. Therefore $|\Phi'(x)| < 1$ in a neighborhood of the solution α. In summary, we obtain

Newton's Method Convergence Criterion. Let f be two times continuously differentiable. If

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

for all x such that $|x-\alpha| \le |x_0-\alpha|$, then Newton's method converges.

Newton's method, provides a way of choosing a Φ suitable for fixed point iteration. There are many other ways to choose Φ as well. In order to compare different fixed point iteration schemes we shall need a way of comparing how fast they converge. Suppose $x_n \to \alpha$ as $n \to \infty$. Let the absolute errors $e_n = x_n - \alpha$. We say the order of convergence of x_n is (at least) p if there exists $K > 0$ such that

$$|e_{n+1}| \le K|e_n|^p \qquad \text{as} \qquad n \to \infty.$$

Note, in the limiting case $p = 1$ we further insist that $K < 1$.

It was shown in the previous lecture that $|\Phi'(x)| \le \lambda < 1$ implies

$$|e_{n+1}| = |\Phi'(c_n)e_n| \le \lambda|e_n|.$$

Therefore, every function Φ that satisfies conditions 1 and 2 has order of convergence at least 1. We now prove that Newton's method has quadratic order of convergence.

Newton's Method Order of Convergence. Suppose f is two times continuously differentiable and let $x = \alpha$ be a solution to $f(x) = 0$. Further suppose $f'(\alpha) \ne 0$. Then, for an initial guess x_0 sufficiently close to α, the fixed point iteration $x_{n+1} = \Phi(x_n)$ given by Newton's method converges to α with order of convergence 2.

By Taylor's theorem there exists c_n between x_n and α such that

$$0 = f(\alpha) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{e_n^2}{2} f''(c_n).$$

By definition

$$e_{n+1} = x_{n+1} - \alpha = \Phi(x_n) - \alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \alpha = e_n - \frac{f(x_n)}{f'(x_n)}.$$

Solving for $f(x_n)/f'(x_n)$ in Taylor's theorem we obtain

$$\frac{f(x_n)}{f'(x_n)} = e_n - \frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)}.$$

Therefore

$$e_{n+1} = e_n - e_n + \frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)} = \frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)}.$$

Letting K be a bound such that

$$\frac{1}{2}\left|\frac{f''(c_n)}{f'(x_n)}\right| \le K \qquad \text{for all} \qquad n \in \mathbf{N},$$

we obtain

$$|e_{n+1}| = \left|\frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)}\right| \le K|e_n|^2.$$

Therefore Newton's method is quadratically convergence.

Note again the proviso that $f'(x) \ne 0$. Nothing good can ever come from dividing by zero and $f'(x)$ appears in the denominator of Newton's method.