MATH/CS 466/666 FALL 2008 QUIZ 4

Key

1. Let A be a invertible $n \times n$ matrix. Define cond(A) the condition number of A.

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$$\text{Cond}(A) = \|A\| \, \|A^{-1}\|$$

2. Let x_a be an approximation of the solution x to $Ax = b$ where A is an $n \times n$ matrix and b is a vector of length n. Define $r = b - Ax_a$. Show that

$$\frac{\|x - x_a\|}{\|x\|} \leq \text{cond}(A) \frac{\|r\|}{\|b\|}.$$

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$\|b\| = \|Ax\| \leq \|A\| \, \|x\|$

Therefore $\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$.

$Ax = b$ so $x = A^{-1}b$

$r = b - Ax_a$ so $x_a = A^{-1}(b - r) = A^{-1}b - A^{-1}r$

Thus,

$$\frac{\|x - x_a\|}{\|x\|} = \frac{\|A^{-1}b - A^{-1}b + A^{-1}r\|}{\|x\|} = \frac{\|A^{-1}r\|}{\|x\|}$$

$$\leq \frac{\|A^{-1}\| \, \|r\|}{\|x\|} \leq \|A^{-1}\| \, \|A\| \frac{\|r\|}{\|b\|} = \text{cond}(A) \frac{\|r\|}{\|b\|}$$

2/2 **3.** Give a simple formula for the sum $\sum_{k=1}^{n-1} k^2$.

Recall that $\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$. Then by the telescoping series trick

$$n^3 - 1 = \sum_{k=1}^{n-1} \left((k+1)^3 - k^3\right) = \sum_{k=1}^{n-1} \left(k^3 + 3k^2 + 3k + 1 - k^3\right) = 3\sum_{k=1}^{n-1} k^2 + 3\sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} 1$$

Thus $\sum_{k=1}^{n-1} k^2 = \frac{n^3-1}{3} - \sum_{k=1}^{n-1} k - \frac{1}{3}\sum_{k=1}^{n-1} 1 = \frac{n^3-1}{3} - \frac{n(n-1)}{2} - \frac{n-1}{3} = \boxed{\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}}$

4. Let A and B be $n \times n$ matrices with entries a_{ij} and b_{ij} respectively. Define $C = AB$. The standard way of computing the elements c_{ij} of C is

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

How many multiplications does it take to fully compute C in this way?

There are n^2 dot products. Each dot product consists of n multiplications. Therefore it takes $\boxed{n^3 \text{ multiplications}}$ to fully compute C in this way.

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5. Let A be an $n \times n$ matrix that can be written as $A = LU$ where L is lower triangular and U is upper triangular. Explain in details the total number of multiplications and divisions generally needed to find L and U using Gauss-Jordan elimination.

Given a $n \times n$ matrix each row operation to eliminate an entry in the first column takes 1 division, $n-1$ multiplications and $n-1$ additions. Each of these row operations must be done $n-1$ times. We then repeat using the $(n-1)\times(n-1)$ matrix in the lower right corner and so on to obtain

divisions: $\overbrace{1\cdot(n-1) + 1\cdot(n-1) + \cdots + 1\cdot(n-1)}^{n-1 \text{ times}} = (n-1)^2$

multiplications: $(n-1)^2 + (n-2)^2 + \cdots + 1^2 = \sum_{k=1}^{n-1} k^2 = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$

additions: $(n-1)^2 + (n-2)^2 + \cdots + 1^2 = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$.

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6. The nodal points x_i and the weights w_i for the Gauss quadrature methods with $n = 2$, 3 and 4 are given in the table

n	x_i	w_i
2	± 0.5773502692	1.0
3	± 0.7745966692	0.5555555556
	0.0	0.8888888889
4	± 0.8611363116	0.3478548451
	± 0.3399810436	0.6521451549

Make the substitution $x = (t-3)/2$ to rewrite the integral

$$\int_1^5 \log(t)\,dt \quad \text{in the form} \quad \int_{-1}^1 f(x)\,dx$$

and then use the Gauss quadrature method with $n = 3$ to approximate this integral.

$2x = t-3$, $t = 2x+3$, $dt = 2dx$ Thus

$\int_1^5 \log(t)\,dt = \int_{-1}^1 2\log(2x+3)\,dx$ so $f(x) = 2\log(2x+3)$

$\sum_{i=1}^3 w_i f(x_i) = (0.55\ldots)f(0.7745\ldots) + (0.88\ldots)f(0.0) + (0.55\ldots)f(-0.7745\ldots)$

$f(0.7745\ldots) = 2\log(2\cdot(0.7745\ldots)+3) \approx 3.0298985796$

$f(0.0) = 2\log 3 \approx 2.1972245734$

$f(-0.7745\ldots) = 2\log(-2\cdot(0.7745\ldots)+3) \approx 0.74423944018$

$\sum_{i=1}^3 w_i f(x_i) \approx 1.6832776985 + 1.953088513 17 + 0.41346635565$

$= 4.0498325676 7$

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7. The Gauss quadrature formula with $n = 4$ is exact for all polynomials of degree less than or equal at most

(A) 7
(B) 13
(C) 14
(D) 27
(E) none of these.

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