Key

MATH/CS 466/666 FALL 2008 EXAM 2

1. Write psuedocode to efficiently evaluate the polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

given a value of x and an array a_i of coefficients as inputs.

```
function y = polyeval(a,x)
    y = a_n
    for k = n-1 down to 0
        y = yx + a_k
    end
```

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2. Consider using the trapeziod method and Simpson's method to approximate

$$\int_1^2 \frac{1}{t}\, dt$$

with $h = (2-1)/n$ where $n = 20$. Without actually performing the computation, tell which method will yeild a better approximation? Explain why in as mathematically precise ~~way~~ as possible? a way that is?

3/3 Simpsons method will be more exact because it is $O(h^4)$ whereas trapezoid method is $O(h^2)$.

Since $h = \frac{1}{20}$ then $h^2 = \frac{1}{400}$ and $h^4 = \frac{1}{160000}$.

Even though the constant in the error estimate of simpsons method may be bigger, possibly 12 times in this case, the fact that h^4 is so much smaller than h^2 implies Simpsons method will yield a better approximation.

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3. The nodal points x_i and the weights w_i for the Gauss quadrature methods with $n = 2$, 3 and 4 are given in the table

n	x_i	w_i
2	±0.5773502692	1.0
3	±0.7745966692	0.5555555556
	0.0	0.8888888889
4	±0.8611363116	0.3478548451
	±0.3399810436	0.6521451549

Make the substitution $x = 2t - 3$ to rewrite the integral

$$\int_1^2 \frac{1}{t}\,dt \quad \text{in the form} \quad \int_{-1}^1 f(x)\,dx$$

and then use the Gauss quadrature method with $n = 3$ to approximate this integral.

$\int_1^2 \frac{dt}{t} = \int_{-1}^1 \frac{2}{x+3}\frac{dx}{2} = \int_{-1}^1 \frac{dx}{x+3}$ so $f(x) = \frac{1}{x+3}$

$x = 2t-3$

$t = \frac{x+3}{2}$ $dt = dx/2$

$\int_{-1}^1 f(x)dx \approx (.55\ldots)f(.774\ldots) + (0.88\ldots)f(0) + (.55\ldots)f(-.774\ldots)$

$= \frac{.555}{.774\ldots+3} + \frac{.888\ldots}{3} + \frac{.555\ldots}{-.774\ldots+3}$ 3/3

$= 0.14718\ldots + 0.29629\ldots + 0.24964$

$= 0.6931216931 5$

4. The Gauss quadrature formula with $n = 7$ is exact for all polynomials of degree at most

(A) 7
(B) 13
(C) 14
(D) 27
(E) none of these.

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5. Use Taylor's theorem to estimate the mathematical error in the approximation

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

$$f(x+h) = \cancel{f(x)} + hf'(x) + \cancel{\frac{h^2}{2} f''(x)} + \frac{h^3}{6} f'''(c_1)$$ for c_1 between x and $x+h$.

$$f(x-h) = \cancel{f(x)} - hf'(x) + \cancel{\frac{h^2}{2} f''(x)} - \frac{h^3}{6} f'''(c_2)$$ and c_2 between x and $x-h$.

Thus $$\frac{f(x+h)-f(x-h)}{2h} = \frac{2hf'(x) + \frac{h^3}{6}(f'''(c_1)+f'''(c_2))}{2h}$$

$$= f'(x) + \frac{h^2}{12}(f'''(c_1)+f'''(c_2)).$$ Thus

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$$\left| f'(x) - \frac{f(x+h)-f(x-h)}{2h} \right| \le \frac{h^2}{6} M \quad \text{where } M = \max_{t\in[x-h,x+h]} |f'''(t)|.$$

6. Explain how roundoff error implies a minimum optimal size for h and how many significant digits are lost by the subtraction of two nearly equal numbers in the numerator of the above formula when performing numerical differentiation.

Suppose f is computed to k significant digits. Then $\text{Rel}\, f_A(x) \approx 5\times 10^{-k}$ and if h is small enough that $|f(t)| \le 2|f(x)|$ for all $|t-x|<h$ then

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$$E = \left| \frac{f_A(x+h) - f_A(x-h)}{2h} - f'(x) \right| \le \frac{4|f(x)|\,5\times 10^{-k}}{2h} + \frac{h^2}{6}M = \frac{\varepsilon}{h} + \frac{h^2}{6}M.$$

minimizing implies $-\frac{\varepsilon}{h^2} + \frac{h}{3}M = 0$ so $h = \sqrt[3]{\frac{3\varepsilon}{M}}$. Thus.

$$E = \varepsilon\sqrt[3]{\frac{M}{3\varepsilon}} + \left(\sqrt[3]{\frac{3\varepsilon}{M}}\right)^2 \frac{1}{6}M = \frac{\sqrt[3]{3}}{6} M^{1/3}\varepsilon^{2/3} = \frac{\sqrt[3]{3}}{6} M^{1/3}(20)^{2/3}|f(x)|^{2/3}\, 10^{-2k/3},$$

Therefore about 1/3 of the significant digits are lost.

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7. Let f be a differentiable function defined on $\mathbf{R}$ with root α. Given an initial guess of x_0 state Newton's method for approximating α.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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and iterate.

(i) Suppose f is twice continuously differentiable and $f'(\alpha) \neq 0$. Show that Newton's method is quadratically convergent.

$$0 = f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(c_n)$$

divide by $f'(x_n)$ to obtain

$$0 = \frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) + \frac{(\alpha - x_n)^2}{2} \frac{f''(c_n)}{f'(x_n)}$$

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Thus

$$\alpha - x_{n+1} = \alpha - x_n + \frac{f(x_n)}{f'(x_n)} = -\frac{(\alpha - x_n)^2}{2} f''(c_n)$$

Defining $E_n = |\alpha - x_n|$ and $M = \max_{t \in [\alpha-h, \alpha+h]} |f''(t)| \Big/ 2 \min_{t \in [\alpha-h, \alpha+h]} |f'(t)|$

where $h>0$ we obtain nearby the root

that $E_{n+1} \leq M E_n^2$ which is quadratic convergence.

(ii) Suppose that α is a root of multiplicity 3 so that $f'(\alpha) = 0$ and $f''(\alpha) = 0$. In this case the quotients

$$\lambda_n = \frac{x_{n+1} - x_n}{x_n - x_{n-1}} \to \lambda = \frac{m-1}{m}$$

will converge to

(A) 1/2

(B) 2/3

(C) 1/3

(D) 3/2

(E) none of these.

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