Near Optimal Approximation with Chebyshev Polynomials

This programming assignment is intended for Math/CS 666 graduate students. Please work individually and not in groups. Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output. If you are an undergraduate you may do this assignment for extra credit.

1a. Consider approximating the function f defined by

$$f(x) = \frac{1}{x^2+1}$$

on the interval $[-2,2]$. Use Newton's divided difference formula to construct the interpolating polynomial $p(x)$ passing through the nine points $(x_n, f(x_n))$ for $n = 0, 1, \ldots 8$ with equally spaced x-coordinates given by $x_n = -2 + n/2$. Plot $f(x)$ and $p(x)$ on the same graph over the interval $[-2,2]$.

1b. Find the coefficents a_i for the Chebyshev polynomial

$$T_9(x) = a_9x^9 + a_8x^8 + \cdots + a_1x + a_0$$

of degree 9. Write a program that uses synthetic division to evaluate this polynomial. Compute the value of $T_9(\sqrt{2})$.

1c. The roots of $T_9(x)$ are given by $r_n = \cos((2n+1)\pi/18)$. Verify these are the roots by computing $T_9(r_n)$ for $n = 0, 1, \ldots, 8$.

1d. Construct the interpolating polynomial $q(x)$ passing through the nine points $(z_n, f(z_n))$ for $n = 0, 1, \ldots, 8$ where $z_n = 2r_n$ are the roots of $T_9(x)$ rescaled to $[-2,2]$. Plot $f(x)$ and $q(x)$ on the same graph.

1e. Let $P(x) = (x-x_0)(x-x_1)\cdots(x-x_8)$ and $Q(x) = (x-z_0)(x-z_1)\cdots(x-z_8)$ where x_n and z_n are as in parts 1a and 1d. Define

$$N(g) = \max\{\, |g(x)| : x \in [-2,2] \,\}$$

and compute $N(P)$, $N(Q)$ and the ratio $\rho_1 = N(Q)/N(P)$.

1f. Plot $f(x)-p(x)$ and $f(x)-q(x)$ on the same graph. Compute $N(f-p)$, $N(f-q)$ and the ratio $\rho_2 = N(f-q)/N(f-p)$.

1g. In what way is ρ_1 related to ρ_2 theoretically? To what extent does this relation hold true in this example?