HW#1

§1.1#14 for $f(x) = e^x$ construct a cubic polynomial $q(x)$ for which

$$q(0) = f(0) \qquad q(1) = f(1)$$
$$q'(0) = f'(0) \qquad q'(1) = f'(1)$$

Numerically compare it to e^x and the Taylor polynomial $p_3(x)$ for $0 \le x \le 1$.

$$q(x) = ax^3 + bx^2 + cx + d$$

$$q'(x) = 3ax^2 + 2bx + c$$

Thus $q(0) = d = f(0) = e^0 = 1$
$q'(0) = c = f'(0) = e^0 = 1$

$q(1) = a + b + 2 = f(1) = e^1 = e$
$q'(1) = 3a + 2b + 1 = f'(1) = e$

Thus $a - 3 = -e \qquad a = 3 - e$
$b + 5 = 2e \qquad b = 2e - 5$

Therefore,

$$q(x) = (3 - e)x^3 + (2e - 5)x^2 + x + 1$$

The Taylor's polynomial is

$$p(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

The accuracy of $q(x)$ is good to 2 decimal digits and accuracy of $p(x)$ is good to 1 decimal digit over the interval $[0,1]$.

§1.1 #4 continues...

```
> restart;
> e:=exp(1);
```

$$e := \mathbf{e}$$

```
> q:=x->(((3-e)*x+(2*e-5))*x+1)*x+1;
  p:=x->(((1/6)*x+1/2)*x+1)*x+1;
  sort(collect(q(x),x));
  sort(collect(p(x),x));
```

$$q := x \to (((3 - e)\, x + 2\, e - 5)\, x + 1)\, x + 1$$

$$p := x \to \left(\left(\frac{1}{6}\, x + \frac{1}{2}\right) x + 1\right) x + 1$$

$$(3 - \mathbf{e})\, x^3 + (-5 + 2\,\mathbf{e})\, x^2 + x + 1$$

$$\frac{1}{6} x^3 + \frac{1}{2} x^2 + x + 1$$

```
> Digits:=32;
  h:=0.1;
  printf(" %4s %20s %20s\n","xn","yn-qn","yn-pn");
  for i from 0 to 10
  do
       xn:=i*h;
       qn:=evalf(q(xn));
       pn:=p(xn);
       yn:=exp(xn);
       printf(" %4g %20.15e %20.15e\n",xn,yn-qn,yn-pn);
  end:
```

$$Digits := 32$$

$$h := 0.1$$

xn	yn-qn	yn-pn
0	0.000000000000000e+00	0.000000000000000e+00
0.1	5.235633349257653e-04	4.251408980958145e-06
0.2	1.686466511118577e-03	6.942482683650059e-05
0.3	2.961687821769183e-03	3.588075760031040e-04
0.4	3.944549555754738e-03	1.158030974603651e-03
0.5	4.365585027986184e-03	2.887937366794814e-03
0.6	4.104758847150176e-03	6.118800390508975e-03
0.7	3.207182742064707e-03	1.158604080380985e-02
0.8	1.900484235920864e-03	2.020759515913427e-02
0.9	6.140019999403591e-04	3.310311115694966e-02
1	0.000000000000000e+00	5.161516179237857e-02

```
>
```

§1.2#2 Find the 2nd degree Taylor polynomial for $f(x) = e^x \sin x$ about the point $a=0$. Bound the error in this approximation when $-\pi/4 \le x \le \pi/4$.

Multiplying series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} e^{c_1}$$

$$\sin x = x - \frac{x^3}{3!} \cos c_2$$

Best bound:

$$\max_{x \in [-\frac{\pi}{4}, \frac{\pi}{4}]} |f(x) - (x + x^2)|$$

$$= \left| f(-\tfrac{\pi}{4}) - \left(-\tfrac{\pi}{4} + (-\tfrac{\pi}{4})^2\right) \right|$$

$$\le 0.1539$$

Thus

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} e^{c_1}\right)\left(x - \frac{x^3}{3!} \cos c_2\right)$$

$$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} e^{c_1} - \frac{x^3}{6} \cos c_2 - \frac{x^4}{6} \cos c_2$$

$$- \frac{x^5}{12} \cos c_2 - \frac{x^6}{36} e^{c_1} \cos c_2$$

The 2nd degree Taylor polynomial is $x + x^2$.

$$|\text{Error}| = |e^x \sin x - (x + x^2)|$$

$$\le \left| x^3 \left(\frac{1}{2} - \frac{1}{3} \cos c_2\right) \right| + \left| \frac{x^4}{6} \cos c_2 \right| + \left| \frac{x^5}{12} \cos c_2 \right|$$

$$+ \left| \frac{x^6}{36} e^{c_1} \cos c_2 \right|$$

$$\le \left(\frac{\pi}{4}\right)^3 \left(\frac{1}{2} - \frac{1}{3} \cos \frac{\pi}{4}\right) + \left(\frac{\pi}{4}\right)^4 \frac{1}{6} + \left(\frac{\pi}{4}\right)^5 \frac{1}{12} + \left(\frac{\pi}{4}\right)^6 \frac{1}{36} e^{\pi/4}$$

$$\approx 0.128045 + 0.063417 + 0.024904 + 0.014300$$

$$\le 0.230667$$

Alt: $|R_2(x)| = \left| \frac{x^3}{3!} 2e^c (\cos c - \sin c) \right| \le \frac{\left(\frac{\pi}{4}\right)^3}{3!} 2 \le 0.1615$

§1.2 #4(a) Bound the error in the approximation $\sin x \approx x$ for $-\pi/4 \le x \le \pi/4$.

$$\sin x = x - \frac{x^3}{3!}\cos c.$$

So

$$|\text{Error}| = |\sin x - x| = \left|\frac{x^3}{3!}\cos c\right|$$

$$\le \left(\frac{\pi}{4}\right)^3 \frac{1}{6} \le 0.080746$$

(b) Since this is a good approximation for small values of x, also consider the percentage error

$$\frac{\sin x - x}{\sin x} \approx \frac{\sin x - x}{x}$$

Bound the absolute value of the later quantity for $-\delta \le x \le \delta$. Pick δ to make the absolute value of the percentage error less than 1%.

$$\left|\frac{\sin x - x}{x}\right| = \left|\frac{\frac{x^3}{3!}\cos c}{x}\right| \le \frac{\delta^2}{6}$$

For less that 1% we obtain

$$\frac{\delta^2}{6} \le .01 \quad \text{so} \quad \delta = 0.244948$$

§1.2#5 How large should the degree $2n-1$ be chosen in (1.14) to have

$$|\sin x - p_{2n-1}(x)| \le 0.001$$

for all $-\pi/2 \le x \le \pi/2$? Check your result by evaluating the resulting $p_{2n-1}(x)$ at $x=\pi/2$.

$$|\sin x - p_{2n-1}(x)| = \left| \frac{x^{2n+1}}{(2n+1)!} \cos c \right|$$

$$\le \left(\frac{\pi}{2}\right)^{2n+1} \frac{1}{(2n+1)!}$$

Table of values (see maple worksheet)

n	$(\pi/2)^{2n+1}/(2n+1)!$
0	1.570797
1	0.645968
2	0.079693
3	0.0046818
4	0.0001605

Thus $n=4$ works.

$$p_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$p_7(\pi/2) = 0.9998431019$$

$$|\sin(\pi/2) - p_7(\pi/2)| \le 0.0001569$$

```
> restart;
```

Problem §1.2#5 Maple worksheet

```
> e:=n->(Pi/2)^(2*n+1)/(2*n+1)!;
```

$$e := n \to \frac{\left(\frac{1}{2}\pi\right)^{(2\,n+1)}}{(2\,n+1)!}$$

```
> for n from 0 to 5
  do
     'n'=n, 'E'<=evalf(e(n));
  end;
```

$$n = 0, E \leq 1.570796327$$

$$n = 1, E \leq 0.6459640976$$

$$n = 2, E \leq 0.07969262631$$

$$n = 3, E \leq 0.004681754139$$

$$n = 4, E \leq 0.0001604411850$$

$$n = 5, E \leq 0.000003598843241$$

```
> p7:=x->x-x^3/3!+x^5/5!-x^7/7!;
```

$$p7 := x \to x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

```
> sin(Pi/2)-evalf(p7(Pi/2));
```

$$0.0001568981$$

```
> evalf(p7(Pi/2));
```

$$0.9998431019$$

```
>
```

§1.3 #5 The error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

is useful in the theory of probability. Find its Taylor polynomial so that the error is bounded by 10^{-9} for $|x| \le b$ for a given $b > 0$. Show how to evaluate the Taylor polynomial efficiently. Draw a graph of the polynomial on $[-b, b]$. Use the values $b = 0.1, 0.3, 0.5$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^c$$

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots + \frac{(-t^2)^n}{n!} + \frac{(-t^2)^{n+1}}{(n+1)!} e^c$$

where $-t^2 \le c \le 0$.

Thus the 2n degree Taylor polynomial gives

$$erf(x) \approx \frac{2}{\sqrt{\pi}} \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots + \frac{(-t^2)^n}{n!}\right) dt$$

$$= \frac{2}{\sqrt{\pi}} \left(t - \frac{t^3}{3} + \frac{t^5}{5 \cdot 2!} - \frac{t^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1) \cdot n!} \right) \Bigg|_0^x$$

$$= \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot n!} \right)$$

§1.3#5 continues...

To evaluate this polynomial efficiently note that it contains only odd powers of x. Thus it can be written as $x\,p(x^2)$ where p is a polynomial of degree n. Let $p(s) = p_0 + p_1 s + \cdots + p_n s^n$.

Now set $s = x^2$ and evaluate $p(s)$ using nested multiplication or equivalently synthetic division. Take the result and multiply by x.

Since $-t^2 \le c \le 0$ then $|e^c| \le 1$. Thus

$$|\text{Error}| \le \frac{2}{\sqrt{\pi}} \left| \int_0^b \frac{t^{2(n+1)}}{(n+1)!}\, dt \right|$$

$$\le \frac{2}{\sqrt{\pi}} \frac{b^{2n+3}}{(2n+3)(n+1)!} \le 10^{-9}$$

All values rounded up for upper bounds give

n	$b=0.1$	$b=0.3$	$b=0.5$
0	0.00038	0.011	0.048
1	0.0000012	0.00028	0.0036
2	2.7×10^{-9}	0.0000059	0.00021
3	5.3×10^{-12}	1.1×10^{-7}	0.000011
4		1.6×10^{-9}	4.2×10^{-7}
5		2.0×10^{-11}	1.5×10^{-8}
6			4.6×10^{-10}
7			1.3×10^{-11}

Therefore $n=3$ works for $b=0.1$; $n=5$ works for $b=0.3$; and $n=7$ works for $b=0.5$.

```
> restart;
```
§1.3#5 continues...

```
> e:=(n,b)->2/sqrt(Pi)*b^(2*n+3)/(2*n+3)/(n+1)!;
```

$$e := (n, b) \rightarrow \frac{2\, b^{(2\,n+3)}}{\sqrt{\pi}\,(2\,n+3)\,(n+1)!}$$

```
> for n from 0 to 7
  do
     'n'=n, 'E'<=evalf(e(n,0.1)),evalf(e(n,0.3)),evalf(e(n,0.5));
  end;
```

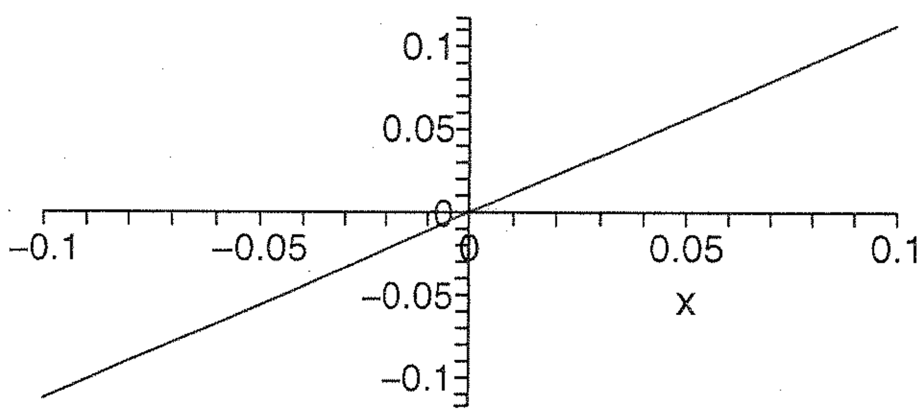
$$n = 0,\ E \le 0.0003761263890,\ 0.01015541250,\ 0.04701579863$$
$$n = 1,\ E \le 0.000001128379167,\ 0.0002741961376,\ 0.003526184897$$
$$n = 2,\ E \le 2.686617065\,10^{-9},\ 0.000005875631517,\ 0.0002098919582$$
$$n = 3,\ E \le 5.223977624\,10^{-12},\ 1.028235516\,10^{-7},\ 0.00001020308130$$
$$n = 4,\ E \le 8.548327022\,10^{-15},\ 1.514310488\,10^{-9},\ 4.173987804\,10^{-7}$$
$$n = 5,\ E \le 1.205533298\,10^{-17},\ 1.922009465\,10^{-11},\ 1.471598264\,10^{-8}$$
$$n = 6,\ E \le 1.492565036\,10^{-20},\ 2.141667688\,10^{-13},\ 4.554947008\,10^{-10}$$
$$n = 7,\ E \le 1.646211436\,10^{-23},\ 2.125920133\,10^{-15},\ 1.255959653\,10^{-11}$$

```
> x3:=expand(2/sqrt(Pi)*(1-s/3+s^2/5/2-s^3/7/3!),s);
  p3:=unapply(x3,s);
```

$$x3 := \frac{2}{\sqrt{\pi}} - \frac{2\,s}{3\,\sqrt{\pi}} + \frac{s^2}{5\,\sqrt{\pi}} - \frac{s^3}{21\,\sqrt{\pi}}$$

$$p3 := s \rightarrow \frac{2}{\sqrt{\pi}} - \frac{2\,s}{3\,\sqrt{\pi}} + \frac{s^2}{5\,\sqrt{\pi}} - \frac{s^3}{21\,\sqrt{\pi}}$$

```
> plot(x*p3(x^2),x=-0.1..0.1);
```

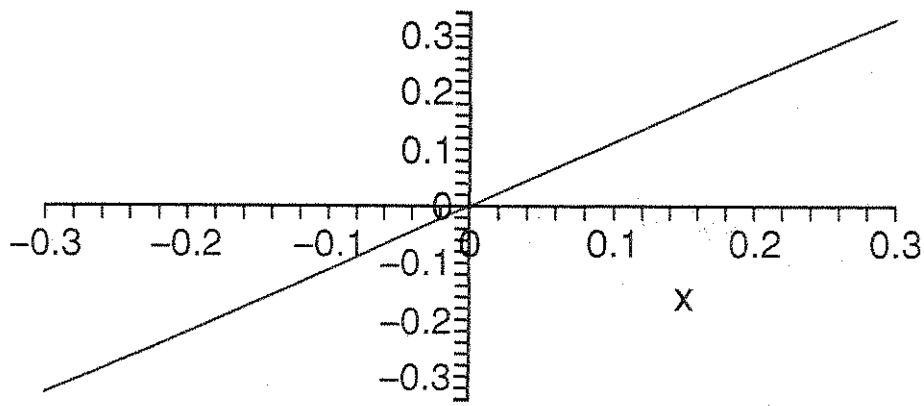


```
> x5:=expand(2/sqrt(Pi)*(1-s/3+s^2/5/2-s^3/7/3!+s^4/9/4!-s^5/11/5!),s);
  p5:=unapply(x5,s);
```

$$x5 := \frac{2}{\sqrt{\pi}} - \frac{2\,s}{3\sqrt{\pi}} + \frac{s^2}{5\sqrt{\pi}} - \frac{s^3}{21\sqrt{\pi}} + \frac{s^4}{108\sqrt{\pi}} - \frac{s^5}{660\sqrt{\pi}}$$

$$p5 := s \rightarrow \frac{2}{\sqrt{\pi}} - \frac{2\,s}{3\sqrt{\pi}} + \frac{s^2}{5\sqrt{\pi}} - \frac{s^3}{21\sqrt{\pi}} + \frac{s^4}{108\sqrt{\pi}} - \frac{s^5}{660\sqrt{\pi}}$$

```
> plot(x*p5(x^2),x=-0.3..0.3);
```

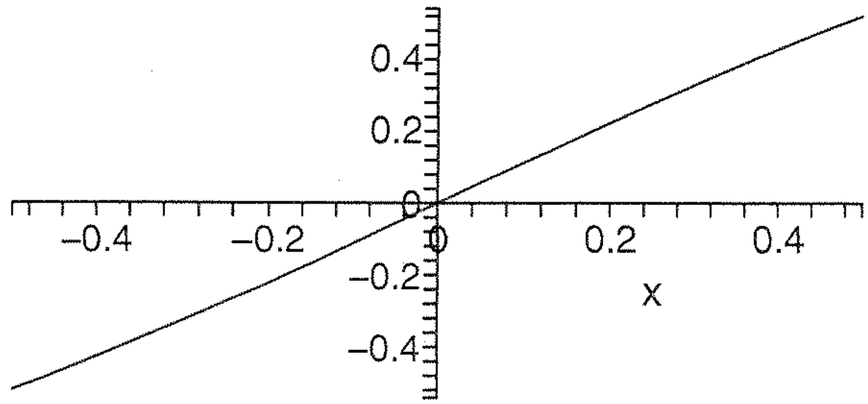



```
> x7:=expand(2/sqrt(Pi)*(1-s/3+s^2/5/2-s^3/7/3!+s^4/9/4!-s^5/11/5!+s^6/13/6
  -s^7/13/6!),s);
  p7:=unapply(x7,s);
```

$$x7 := \frac{2}{\sqrt{\pi}} - \frac{2\,s}{3\sqrt{\pi}} + \frac{s^2}{5\sqrt{\pi}} - \frac{s^3}{21\sqrt{\pi}} + \frac{s^4}{108\sqrt{\pi}} - \frac{s^5}{660\sqrt{\pi}} + \frac{s^6}{4680\sqrt{\pi}} - \frac{s^7}{4680\sqrt{\pi}}$$

$$p7 := s \rightarrow \frac{2}{\sqrt{\pi}} - \frac{2\,s}{3\sqrt{\pi}} + \frac{s^2}{5\sqrt{\pi}} - \frac{s^3}{21\sqrt{\pi}} + \frac{s^4}{108\sqrt{\pi}} - \frac{s^5}{660\sqrt{\pi}} + \frac{s^6}{4680\sqrt{\pi}} - \frac{s^7}{4680\sqrt{\pi}}$$

```
> plot(x*p7(x^2),x=-0.5..0.5);
```


0.4
0.2
0
-0.2
-0.4
-0.4
-0.2
0
0.2
0.4
x

>

§1.3 Evaluate

$$p(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \frac{x^{12}}{12!} - \frac{x^{15}}{15!}$$

as efficiently as possible.

Note that $p(x)$ may be written as $q(x^3)$ where $q(s)$ is the polynomial

$$q(s) = 1 - \frac{s}{3!} + \frac{s^2}{6!} - \frac{s^3}{9!} + \frac{s^4}{12!} - \frac{s^5}{15!}$$

Now it takes two multiplications to form $s = x^3$ from x. Then it takes 5 more multiplications and 5 additions to evaluate $q(s)$ by nested multiplication or synthetic division. The total number of operations is

multiplications 7
additions 5

§1.3 #10 Show how to evaluate the function

$$f(x) = 2e^{4x} - e^{3x} + 5e^{x} + 1$$

efficiently. Hint $z = e^x$.

Note that $f(x) = p(e^x)$ where $p(z)$ is a polynomial of degree 4 in z given by

$$p(z) = 2z^4 - z^3 + 5z + 1$$

Now evaluate $p(z)$ using synthetic division or nested multiplication. The total number of operations are

transcendental	1
multiplications	4
additions	3