MATH/CS 466/666 HW #4

§4.1 #15* Find a polynomial $P(x)$ of degree ≤ 3 for which

$P(0) = y_0$ $\quad P(1) = y_2$

$P'(0) = y_1'$ $\quad P'(1) = y_2'$

with y_1, y_2, y_1' and y_2' given constants. The resulting polynomial is called the cubic Hermite interpolating polynomial.

First, we find functions H_1, H_2, H_3 and H_4 such that

$H_1(0) = 1 \quad H_1'(0) = H_1(1) = H_1'(1) = 0$

$H_2(0) = 0 \quad H_2'(0) = 1 \quad H_2(1) = H_2'(1) = 0$

$H_3(0) = H_3'(0) = 0 \quad H_3(1) = 1 \quad H_3'(1) = 0$

$H_4(0) = H_4'(0) = H_4(1) = 0 \quad H_4'(1) = 1.$

First H_1: $\quad H_1(x) = (x-1)^2(ax+b)$ satisfies $H_1(1) = H_1'(1) = 0.$

Now $H_1'(x) = 2(x-1)(ax+b) + a(x-1)^2$ so

$H_1(0) = b = 1$ so $b = 1$

$H_1'(0) = -2b + a = 0$ so $a = 2.$

$H_1(x) = (x-1)^2(2x+1)$

Second H_3: $\quad H_3(x) = x^2(ax+b)$ satisfies $H_3(0) = H_3'(0) = 0.$

Now $H_3'(x) = 2x(ax+b) + ax^2$ so

$H_3(1) = a+b = 1$ and $H_3'(1) = 3a + 2b = 0$

implies $b = 3$ and $a = -2.$

$H_3(x) = x^2(-2x+3).$

Third H_2: $\quad H_2(x) = x(x-1)(ax+b)$ satisfies $H_2(0) = H_2(1) = 0.$

Now $H_2'(x) = (2x-1)(ax+b) + ax(x-1)$ so

$H_2'(0) = -b = 1$ implies $b = -1.$

$H_2'(1) = a + b = 0$ so $a = 1.$

$H_2(x) = x(x-1)^2$

§4.1 #15* continues...

Fourth H_4: $H_4(x) = x(x-1)(ax+b)$ satisfies $H_4(0) = H_4(1) = 0$

Now $H_4'(x) = (2x-1)(ax+b) + ax(x-1)$ so

$H_4'(0) = -b = 0$ implies $b=0$

$H_4'(1) = a+b = 1$ implies $a=1$

$H_4(x) = x^2(x-1)$.

Therefore.

$$P(x) = y_1 H_1(x) + y_1' H_2(x) + y_2 H_3(x) + y_2' H_4(x)$$

$$= y_1 (x-1)^2(2x+1) + y_1' x(x-1)^2 + y_2 x^2(-2x+3) + y_2' x^2(x-1).$$

§4.1 #28 The following data are taken from a polynomial $p(x)$ of degree ≤ 5. What is the polynomial and what is its degree.

x_n	$p(x_n)$	$p(x_n, x_{n+1})$	$p(x_n, \ldots, x_{n+2})$	$p(x_n, \ldots, x_{n+3})$	$p(x_n, \ldots, x_{n+4})$	$p(x_n, \ldots, x_{n+5})$
-2	-5	6	-3	1	0	0
-1	1	0	0	1	0	
0	1	0	3	1		
1	1	6	6			
2	7	18				
3	25					

Newton's divided difference formula yields

$$p(x) = -5 + (x+2)6 + (x+2)(x+1)(-3) + (x+2)(x+1)x$$

which is degree 3.

§4.1 #31 Let $f(x) = x^n$ for some integer $n \geq 0$. Let $x_0, \ldots, x_m$ be $m+1$ distinct numbers. What is $f(x_0, \ldots, x_m)$ for $m = n$? For $m > n$?

Simple test case $n = 2$. Then

x_n	x_n^2	$f(x_n, x_{n+1})$	$f(x_n, \ldots, x_{n+2})$	$f(x_n, \ldots, x_{n+3})$
0	0	1	1	0
1	1	3	1	
2	4	5		
3	9			

It appears that $f(x_0, \ldots, x_m) = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m > 0 \end{cases}$.

Since $f(x) = x^n$ is a polynomial of degree n and the interpolating polynomial of degree n is unique then

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + \cdots + (x - x_0) \cdots (x - x_{m-1}) f(x_0, \ldots, x_m)$$

for $m \geq n$. Now if $m = n$ then equating powers of x^n we obtain

$$x^n = x^n f(x_0, \ldots, x_n)$$

which implies $f(x_0, \ldots, x_n) = 1$. If $m > n$ then equating the highest power of x^m yields

$$0 = x^m f(x_0, \ldots, x_m)$$

which implies $f(x_0, \ldots, x_m) = 0$ for $m > n$.

§4.1 #32* Let $f(x)=e^x$. Show that $f(x_0,\dots,x_m)>0$ for all values of m and all distinct nodes $x_0,\dots,x_m$.

Since the order of the nodes doesn't matter we assume that $x_0<x_1<\cdots<x_m$. Newton's divided difference formula

$$e^x = f(x_0)+(x-x_0)f(x_0,x_1)+\cdots+(x-x_0)\cdots(x-x_{m-1})f(x_0,\dots,x_{m-1},x)$$

Dividing by e^x and using the fact that $x^n/e^x\to 0$ as $x\to\infty$ for any n we obtain

$$\lim_{x\to\infty}\frac{(x-x_0)\cdots(x-x_{m-1})f(x_0,\dots,x_{m-1},x)}{e^x}=1$$

which implies for x large enough that $f(x_0,\dots,x_{m-1},x)>0$.

Now, since f is continuous, then $f(x_0,\dots,x_{m-1},x)$ is continuous for all $x>x_{m-1}$. If $f(x_0,\dots,x_{m-1},x_m)<0$ then there must be some point $x_*>x_m$ such that $f(x_0,\dots,x_{m-1},x_*)=0$.

This implies the interpolating polynomial $p(x)$ through the nodes $x_0,\dots,x_{m-1},x_*$ is of degree $m-1$. Define $g(x)=e^x-p(x)$.

Now, $g(x)=0$ at the $m+1$ nodes $x_0,\dots,x_{m-1},x_*$ and so by the mean value theorem $g'(x)=0$ for $m-1$ values of x, one between each of the nodes. Similary $g''(x)$ has $m-2$ zero's and finally $g^{(m)}(x)$ has 1 zero.

But then $g^{(m)}(x)=e^x-p^{(m)}(x)=e^x$ because p was a polynomial of degree $m-1$. However $e^x>0$ so $g^{(m)}(x)$ could, in fact, not have any zero's. It follows that $f(x_0,\dots,x_{m-1},x_m)>0$.

§4.2 #9. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial of degree less than or equal n and let $x_0, \ldots, x_n$ be distinct points. What is the value of $f(x_0, \ldots, x_n)$?

Since f is a polynomial and the interpolating polynomial of degree less than or equal n is unique, then

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + \cdots + (x - x_0)\cdots(x - x_{n-1}) f(x_0, \ldots, x_n)$$

Equating powers of x^n yields

$$a_n x^n = x^n f(x_0, \ldots, x_n)$$

Therefore $f(x_0, \ldots, x_n) = a_n$.

§4.2 #11 Consider using the nodes

$$x_0 = a - \frac{h}{\sqrt{3}} \quad \text{and} \quad x_1 = a + \frac{h}{\sqrt{3}}$$

to linearly approximate a function on $[a-h, a+h]$ for some real numbers a and h with $h>0$. Calculate a bound for

$$\max_{a-h \le x \le a+h} |f(x) - p(x)|$$

By Theorem 4.2.1 we have

$$f(x) - p(x) = \frac{(x-x_0)(x-x_1)}{2} f''(c_x)$$

Let $M = \max_{a-h \le x \le a+h} |f''(x)|$. Then

$$|f(x) - p(x)| \le \frac{M}{2} \max_{a-h \le x \le a+h} |(x-x_0)(x-x_1)|$$

Now

$$(x-x_0)(x-x_1) = (x-a+\frac{h}{\sqrt{3}})(x-a-\frac{h}{\sqrt{3}}) = (x-a)^2 - \frac{h^2}{3}$$

Implies the minimum is at $x=a$ and equal $-\frac{h^2}{3}$. The maximum must occur at one of the endpoint of the interval and is $(a+h-a)^2 - \frac{h^2}{3} = \frac{2h^2}{3}$. Therefore the max is $2h^2/3$. Thus

$$\max_{a-h \le x \le a+h} |f(x) - p(x)| \le \frac{M}{3} h^2.$$

§4.2#11 continues....

Illustrate this result for $f(x) = e^x$. Choose values for h and a and then draw a graph to illustrate what you are calculating. This problem has applications in solving differential and integral equations.

$$M = \max_{a-h \le x \le a+h} |f''(x)| = e^{a+h}$$

Thus $$\max_{a-h \le x \le a+h} |f(x) - p(x)| \le \frac{h^2}{3} e^{a+h}.$$

For an example we take $a=0$ and $h=1$. Then

$$p(x) = \frac{e^{-1/\sqrt{3}}(x - 1/\sqrt{3})}{-2/\sqrt{3}} + \frac{e^{1/\sqrt{3}}(x + 1/\sqrt{3})}{2/\sqrt{3}}$$

and $$\max_{1 \le x \le 1} |e^x - p(x)| \le \frac{1}{3} e$$

A plot showing the bound and the actual difference done in Maple is on the following page.

```
> restart;
```

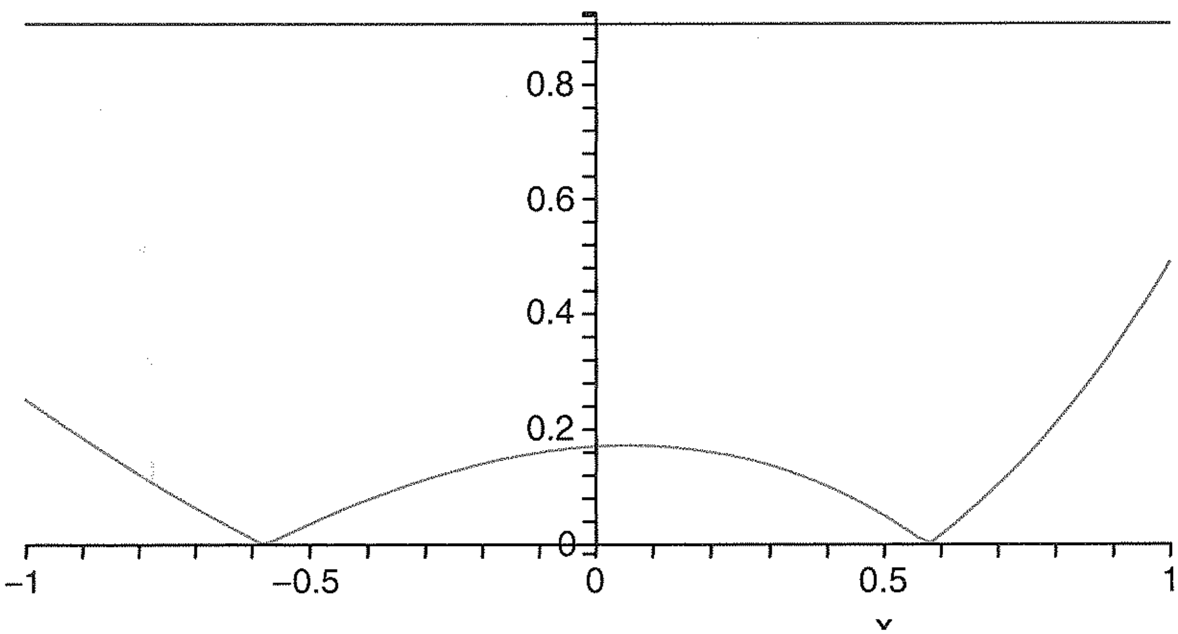
§4.2 #11 continues ...

```
> B:=exp(1)/3;
  p:=x->exp(-1/sqrt(3))*(x-1/sqrt(3))/(-2/sqrt(3))
      +exp(1/sqrt(3))*(x+1/sqrt(3))/(2/sqrt(3));
```

$$B := \frac{1}{3}\,\mathrm{e}$$

$$p := x \to -\frac{1}{2}\,\mathrm{e}^{\left(-\frac{1}{\sqrt{3}}\right)}\left(x-\frac{1}{\sqrt{3}}\right)\sqrt{3}+\frac{1}{2}\,\mathrm{e}^{\left(\frac{1}{\sqrt{3}}\right)}\left(x+\frac{1}{\sqrt{3}}\right)\sqrt{3}$$

```
> plot([B,abs(exp(x)-p(x))],x=-1..1);
```




```
>
```

§4.3# 3a Consider the data

x	0	1/2	1	2	3
y	0	1/4	1	-1	-1

(a) Find the piecewise linear interpolating function for the data.

$$\ell(x) = \begin{cases} \frac{1}{2}x & \text{for } x\in[0,1/2] \\ 2\left(\frac{-1}{4}(x-1) + (x-\frac{1}{2})\right) & \text{for } x\in[1/2,1] \\ -(x-2)-(x-1) & \text{for } x\in[1,2] \\ (x-3)-(x-2) & \text{for } x\in[2,3]. \end{cases}$$

$$= \begin{cases} \frac{1}{2}x & \text{for } x\in[0,1/2] \\ \frac{3}{4}x-\frac{1}{4} & \text{for } x\in[1/2,1] \\ -2x+3 & \text{for } x\in[1,2] \\ -1 & \text{for } x\in[2,3] \end{cases}$$

§4.3 #3 continues...

(c) Find the natural cubic spline.

We set $M_j = S''(x_j)$ for $j = 1, \ldots, 5$ and then solve.

$$\frac{x_j - x_{j-1}}{6} M_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} M_j + \frac{x_{j+1} - x_j}{6} M_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

$M_1 = M_n = 0$ and $j = 2, \ldots, 4$

Recall the data

j	1	2	3	4	5
x_j	0	1/2	1	2	3
y_j	0	1/4	1	-1	-1

and create the tridiagonal matrix

$$\begin{bmatrix} \frac{x_3 - x_1}{3} & \frac{x_3 - x_2}{6} & 0 \\ \frac{x_3 - x_2}{6} & \frac{x_4 - x_2}{3} & \frac{x_4 - x_3}{6} \\ 0 & \frac{x_4 - x_3}{6} & \frac{x_5 - x_3}{3} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/12 & 0 \\ 1/12 & 1/2 & 1/6 \\ 0 & 1/6 & 2/3 \end{bmatrix}$$

and the vector

$$\begin{bmatrix} \frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \\ \frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \\ \frac{y_5 - y_4}{x_5 - x_4} - \frac{y_4 - y_3}{x_4 - x_3} \end{bmatrix} = \begin{bmatrix} \frac{3/4}{1/2} - \frac{1/4}{1/2} \\ \frac{-2}{1} - \frac{3/4}{1/2} \\ 0 - \frac{-2}{1} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{7}{2} \\ 2 \end{bmatrix}$$

§4.3 #3 continues...

Solve for M_2, M_3 and M_4

$$\begin{bmatrix} 1/3 & 1/12 & 0 \\ 1/12 & 1/2 & 1/6 \\ 0 & 1/6 & 2/3 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -7/2 \\ 2 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1/3 & 1/12 & 0 & 1 \\ 1/12 & 1/2 & 1/6 & -7/2 \\ 0 & 1/6 & 2/3 & 2 \end{array}\right] \xrightarrow{r_2 - \frac{1}{4} r_1} \begin{bmatrix} 1/3 & 1/12 & 0 & 1 \\ 0 & 23/48 & 1/6 & -15/4 \\ 0 & 1/6 & 2/3 & 2 \end{bmatrix} \xrightarrow{r_3 - \frac{8}{23} r_2}$$

$$\begin{bmatrix} 1/3 & 1/12 & 0 & 1 \\ 0 & 23/48 & 1/6 & -15/4 \\ 0 & 0 & 14/23 & 76/23 \end{bmatrix}$$

$M_4 = 38/7$

$M_3 = \frac{48}{23}\left(-\frac{15}{4} - \frac{1}{6} \cdot \frac{38}{7}\right) = -\frac{68}{7}$

$M_2 = 3\left(1 + \frac{1}{12} \cdot \frac{68}{7}\right) = \frac{38}{7}$

Now substitute into equation (4.63) see work on attached Maple Worksheet to obtain

$$S(x) = \begin{cases} \frac{38}{21}x^3 + \frac{1}{21}x & \text{if } x \in [0, \frac{1}{2}] \\ -\frac{106}{21}x^3 + \frac{72}{7}x^2 - \frac{107}{21}x + \frac{6}{7} & \text{if } x \in [\frac{1}{2}, 1] \\ \frac{53}{21}x^3 - \frac{87}{7}x^2 + \frac{370}{21}x - \frac{47}{7} & \text{if } x \in [1, 2] \\ -\frac{19}{21}x^3 + \frac{57}{7}x^2 - \frac{494}{21}x + \frac{145}{7} & \text{if } x \in [2, 3] \end{cases}$$

```
> restart;
```
§4.3#36 continues...

```
> # Substitute values for xj, yj and Mj into (4.63)
> sn:=((x[j]-t)^3*M[j-1]+(t-x[j-1])^3*M[j])/6/(x[j]-x[j-1])
    +((x[j]-t)*y[j-1]+(t-x[j-1])*y[j])/(x[j]-x[j-1])
    -1/6*(x[j]-x[j-1])*((x[j]-t)*M[j-1]+(t-x[j-1])*M[j]);
```

$$sn := \frac{(x_j - t)^3 M_{j-1} + (t - x_{j-1})^3 M_j}{6(x_j - x_{j-1})} + \frac{(x_j - t)\, y_{j-1} + (t - x_{j-1})\, y_j}{x_j - x_{j-1}} - \frac{1}{6}(x_j - x_{j-1})\left((x_j - t) M_{j-1} + (t - x_{j-1}) M_j\right)$$

```
> xn:=[0,1/2,1,2,3];
  yn:=[0,1/4,1,-1,-1];
  Mn:=[0,38/7,-68/7,38/7,0];
```

$$xn := \left[\, 0, \frac{1}{2}, 1, 2, 3 \right]$$

$$yn := \left[\, 0, \frac{1}{4}, 1, -1, -1 \right]$$

$$Mn := \left[\, 0, \frac{38}{7}, \frac{-68}{7}, \frac{38}{7}, 0 \right]$$

```
> for i from 2 to 5
  do
     tmp:=subs([j=i,x=xn,y=yn,M=Mn,t=x],sn):
     print(sort(simplify(tmp),x));
  end:
```

$$\frac{38}{21} x^3 + \frac{1}{21} x$$

$$-\frac{106}{21} x^3 + \frac{72}{7} x^2 - \frac{107}{21} x + \frac{6}{7}$$

$$\frac{53}{21} x^3 - \frac{87}{7} x^2 + \frac{370}{21} x - \frac{47}{7}$$

$$-\frac{19}{21} x^3 + \frac{57}{7} x^2 - \frac{494}{21} x + \frac{145}{7}$$

§4.3 #12 Define

$$s(x) = \begin{cases} 2x^3 & x \in [0,1] \\ x^3+3x^2-3x+1 & x \in [1,2] \\ 9x^2-15x+9 & x \in [2,3] \end{cases}$$

Verify that $s(x)$ is a cubic spline on $[0,3]$. Is it a natural cubic spline function on this interval?

To be a cubic spline we need.

1. $s(x)$ is degree ≤ 3 on each subinterval.
2. $s(x)$, $s'(x)$ and $s''(x)$ are continuous.

$$\lim_{x \to 1^-} s(x) = \lim_{x \to 1^-} 2x^3 = 2$$

$$\lim_{x \to 1^+} s(x) = \lim_{x \to 1^+} x^3+3x^2-3x+1 = 1+\cancel{3}-\cancel{3}+1 = 2$$

Therefore $s(x)$ is continuous at $x=1$.

$$\lim_{x \to 2^-} s(x) = \lim_{x \to 2^-} x^3+3x^2-3x+1 = 8+12-6+1 = 15$$

$$\lim_{x \to 2^+} s(x) = \lim_{x \to 2^-} 9x^2-15x+9 = 36-30+9 = 15$$

Therefore $s(x)$ is continuous at $x=2$.

Therefore $s(x)$ is continuous on $[0,3]$.

$$\lim_{x \to 1^-} s'(x) = \lim_{x \to 1^-} 6x^2 = 6$$

§4.3 #12 continues...

$$\lim_{x \to 1^+} s'(x) = \lim_{x \to 1^+} 3x^2+6x-3 = 3+6-3 = 6$$

Therefore $s'(x)$ exists and is continuous at $x=1$.

$$\lim_{x \to 2^-} s'(x) = \lim_{x \to 2^-} 3x^2+6x-3 = 12+12-3 = 21$$

$$\lim_{x \to 2^+} s'(x) = \lim_{x \to 2^+} 18x-15 = 36-15 = 21$$

Therefore $s'(x)$ exists and is continuous at $x=2$.

$$\lim_{x \to 1^-} s''(x) = \lim_{x \to 1^-} 12x = 12$$

$$\lim_{x \to 1^+} s''(x) = \lim_{x \to 1^+} 6x+6 = 12$$

Therefore $s''(x)$ exists and is continuous at $x=1$

$$\lim_{x \to 2^-} s''(x) = \lim_{x \to 2^-} 6x+6 = 18$$

$$\lim_{x \to 2^+} s''(x) = \lim_{x \to 2^+} 18 = 18$$

Therefore $s''(x)$ exists and is continuous at $x=2$.

Obviously $s(x)$ is piecewise polynomial of degree less or equal 3 on each subinterval. It follows that $s(x)$ is a cubic spline function.

§4.3 # 12 continues...

Is it a natural cubic spline function?

$$s''(0) = 12x\Big|_{x=0} = 0$$

$$s''(3) = 18$$

On the right endpoint the second derivative is not zero. Therefore $s(x)$ is not a natural cubic spline function.

§4.4 #2(a) For $f(x) = \tan^{-1}(x)$ calculate the Taylor approximations $t_1(x)$ and $t_3(x)$. Also find their maximum errors as approximations to $\tan^{-1}(x)$ on $[-1, 1]$.

We will take the Taylor series expanded around $x = 0$.

$f(x) = \tan^{-1}(x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $\qquad f''(0) = 0$

$$f'''(x) = \frac{-2(1+x^2)^2 + 2x(2)(1+x^2)2x}{(1+x^2)^4}$$

$f'''(0) = -2$

Thus

$t_1(x) = x$

$t_3(x) = x - \frac{2}{3!}x^3 = x - \frac{1}{3}x^3$

Compute

$$\max_{x \in [-1,1]} |\tan^{-1}(x) - x|$$

we differentiate $\frac{1}{1+x^2} - 1 = 0$ to obtain critical points at $x = 0$.

§4.4#2a continues

Therefore,

$$\max_{x\in[-1,1]} |\tan^{-1}(x)-x|$$

$$=\max\{|\tan^{-1}(-1)+1|, |\tan^{-1}(0)-0|, |\tan^{-1}(1)-1|\}$$

$$=\frac{\pi}{4}-1 \leq 0.214602$$

Compute

$$\max_{x\in[-1,1]} |\tan^{-1}(x)-(x-\tfrac{1}{3}x^3)|$$

Differentiate $\frac{1}{1+x^2}-1+x^2=0$

$$(x^2-1)(x^2+1)=0$$

so critical points are $x=\pm 1$. Therefore by symmetry

$$\max_{x\in[-1,1]} |\tan^{-1}(x)-(x-\tfrac{1}{3}x^3)| = \tan^{-1}(1)-1+\frac{1}{3}$$

$$=\frac{\pi}{4}-\frac{2}{3} \leq 0.118732$$

§4.4 #2b The linear and cubic minimax polynomials for $f(x) = \tan^{-1}(x)$ on $[-1,1]$ are respectively.

$$m_1(x) = 0.833278x$$

$$m_3(x) = 0.97238588x - 0.19193797x^3$$

Find their maximum errors on $[-1,1]$.

For $m_1(x)$ find critical points

$$\frac{1}{1+x^2} - 0.833278 = 0$$

so $x = \pm 0.447303$. Therefore by symmetry

$$\max_{x\in[-1,1]} |f(x) - m_1(x)|$$

$$= \max\{|f(1)-m_1(1)|, |f(0.447303)-m_1(0.447303)|\}$$

$$\le \max\{0.0478798, 0.0478811\} = 0.0478811$$

For m_3 find critical points

$$\frac{1}{1-x^2} - 0.97238588 + (0.19193797)(3)x^2 = 0$$

Let P be the set of critical points union with $\{-1,1\}$. Then

$$\max_{x\in[-1,1]} |f(x)-m_3(x)| = \max_{x\in P} |f(x)-m_3(x)|$$

$$\le 0.00495409$$

Maple worksheet to find the critical points and to compute this maximum follows.

```
> # S4.4#2b Compute critical points for error estimates on m3 min-max polynomial
> restart;
> f:=x->arctan(x);
m3:=x->0.97238588*x-0.19193797*x^3;
```

$$f := x \rightarrow \arctan(x)$$

$$m3 := x \rightarrow 0.97238588\, x - 0.19193797\, x^3$$

```
> tmp:=diff(f(x)-m3(x),x);
```

$$tmp := \frac{1}{1+x^2} - 0.97238588 + 0.57581391\, x^2$$

```
> cp:=solve(tmp=0,x);
```

$$cp := -0.7810969900, 0.7810969900, -0.2803622674, 0.2803622674$$

```
> p:=[cp,1,-1];
```

$$p := [-0.7810969900, 0.7810969900, -0.2803622674, 0.2803622674, 1, -1]$$

```
> n:=nops(p);
for i from 1 to n
do
    evalf(abs(f(p[i])-m3(p[i])));
end;
```

$$n := 6$$

0.0049503234

0.0049503234

0.0049540894 ← largest one

0.0049540894

0.0049502535

0.0049502535

```
>
```

§4.4 #2c Graph $f(x)-t_3(x)$ and $f(x)-m_3(x)$ on $[-1,1]$.

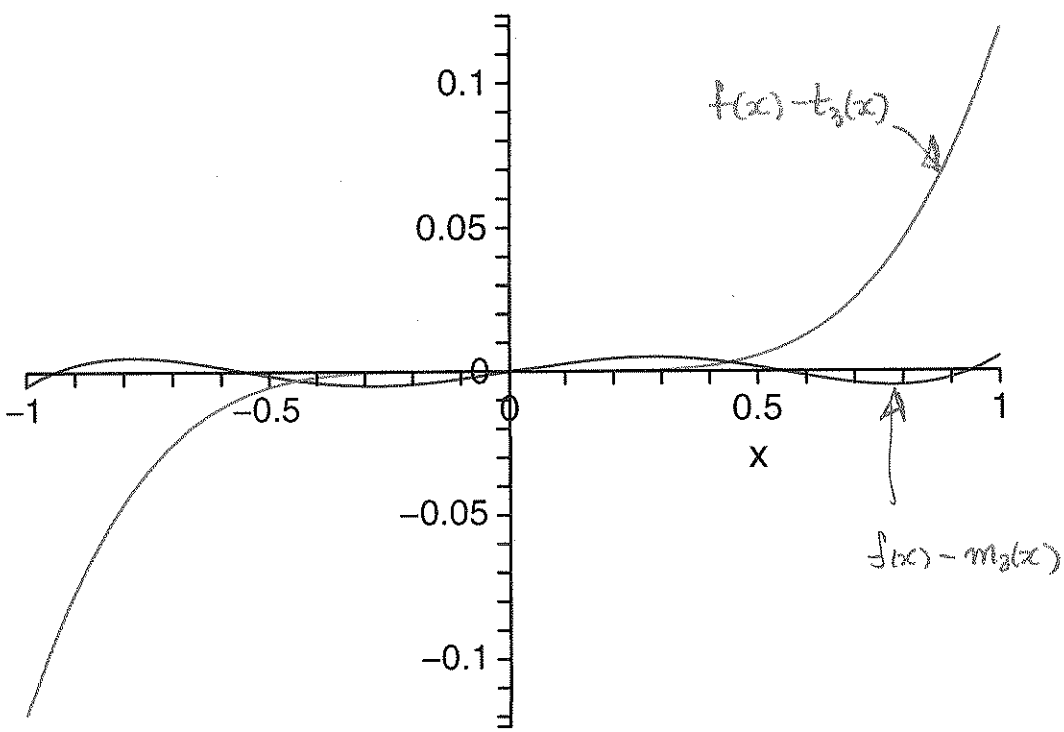
```
> # S4.4#2c Graph f-t3 and f-m3 on [-1,1]
> restart;
> f:=x->arctan(x);
  t3:=x->x-x^3/3;
  m3:=x->0.97238588*x-0.19193797*x^3;
```

$$f := x \to \arctan(x)$$

$$t3 := x \to x - \frac{1}{3}x^3$$

$$m3 := x \to 0.97238588\,x - 0.19193797\,x^3$$

```
> plot({f(x)-t3(x),f(x)-m3(x)},x=-1..1);
```



§4.4#5 Compute the bound

$$\rho_n(f) \le \frac{[(b-a)/2]^{n+1}}{(n+1)!\,2^n} \max_{x\in[a,b]} |f^{(n+1)}(x)|$$

for $f(x) = \cos(x)$ on the interval $[0, \pi/2]$ for $n = 1, 2, \ldots, 7$.

Since maximum of $|\sin(x)|$ and $|\cos(x)|$ are both one on this interval, then the bound may be computed as

$$\frac{(\pi/4)^{n+1}}{(n+1)!\,2^n}$$

```
> # S4.4p5a Computing the bound
> restart;
> b:=n->(Pi/4)^(n+1)/(n+1)!/2^n;
```

$$b := n \to \frac{\left(\frac{1}{4}\pi\right)^{(n+1)}}{(n+1)!\,2^n}$$

```
> for i from 1 to 7
  do
     rho[i](f) <= evalf(b(i));
  end;
```

$$\rho_1(f) \le 0.1542125688$$
$$\rho_2(f) \le 0.02018637805$$
$$\rho_3(f) \le 0.001981793031$$
$$\rho_4(f) \le 0.0001556496608$$
$$\rho_5(f) \le 0.00001018724647$$
$$\rho_6(f) \le 5.715031909\,10^{-7}$$
$$\rho_7(f) \le 2.805359728\,10^{-8}$$

```
>
```

§4.4 #5 To check the accuracy of (a) compute the exact error in the approximation $m_3(x) \approx \cos(x)$ on $[0, \pi/2]$ where

$m_3(x) = 0.9986329 + 0.0296140x - 0.6008616x^2 + 0.1125060x^3$

From the attached Maple worksheet we see that

$$\max_{x \in [0, \pi/2]} |\cos(x) - m_3(x)| = \max_{x \in P} |\cos(x) - m_3(x)|$$

$$\leq 0.00136713$$

From the bound given in part (a) we had

$$\rho_3(f) \leq 0.00198180$$

Therefore the bound given by the Chebyshev theory is of the same order of magnitude as the exact bound for the real min-max polynomial.

```
> # S4.4p5b Find the error for m3 min-max polynomial
> restart;
> f:=x->cos(x);
  m3:=x->0.9986329+0.0296140*x-0.6008616*x^2+0.1125060*x^3;
```

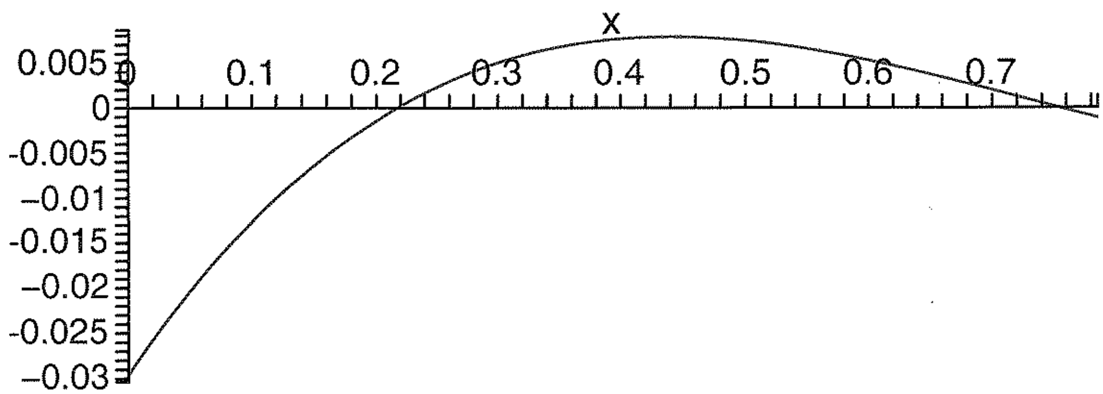
$$f := x \to \cos(x)$$

$$m3 := x \to 0.9986329 + 0.0296140\,x - 0.6008616\,x^2 + 0.1125060\,x^3$$

```
> tmp:=diff(f(x)-m3(x),x);
```

$$tmp := -\sin(x) - 0.0296140 + 1.2017232\,x - 0.3375180\,x^2$$

```
> plot(tmp,x=0..Pi/4);
```




```
> cp1:=fsolve(tmp=0,x=0.2);
  cp2:=fsolve(tmp=0,x=0.7);
```

$$cp1 := 0.2174377516$$

$$cp2 := 0.7551284367$$

```
> p:=[cp1,cp2,0,Pi/4]:
> n:=nops(p):
  for i from 1 to n
  do
      evalf(abs(f(p[i])-m3(p[i])));
  end;
```

0.0013670499

0.0013671259 ← largest one

0.0013671

0.00135061552

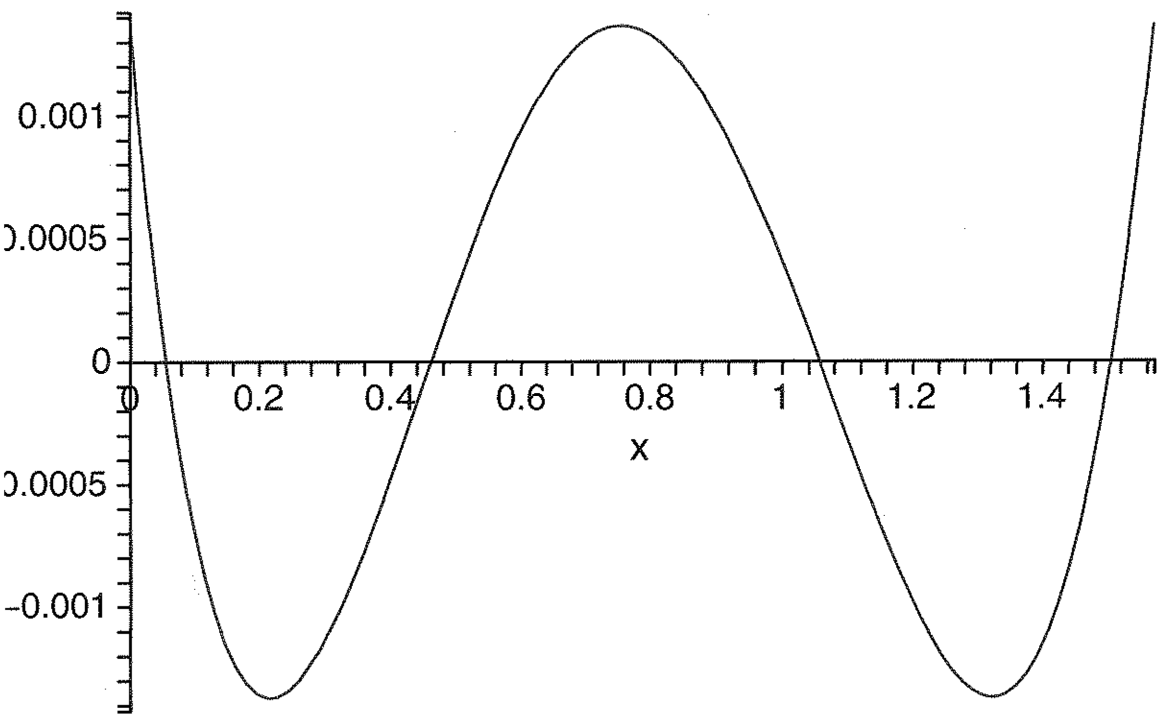
```
>
```

```
> # S4.4p5c Graph cos(x)-m3(x) on [0,Pi/2]
> restart;
> f:=x->cos(x);
  m3:=x->0.9986329+0.0296140*x-0.6008616*x^2+0.1125060*x^3;
```

$$f := x \rightarrow \cos(x)$$

$$m3 := x \rightarrow 0.9986329 + 0.0296140\,x - 0.6008616\,x^2 + 0.1125060\,x^3$$

```
> plot(f(x)-m3(x),x=0..Pi/2);
```




```
>
```

§4.5 #4.5 #6 Let $q(x)$ be a polynomial of degree $\le n-1$ and consider

$$\max_{x\in[-1,1]} |x^n - q(x)|$$

What is the smallest possible value for this quantity? Solve for the $q(x)$ for which this smallest value is attained.

We know that among all monic polynomials of degree n that the modified Chebyshev polynomial has the smallest absolute value. Thus

$$\frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |\tilde{T}_n(x)| = \min_{\substack{p \text{ monic} \\ \text{and} \\ \deg(p)\le n.}} \max_{x\in[-1,1]} |p(x)|$$

Since any monic polynomial of degree n can be written as $x^n - q(x)$ where $q(x) = x^n - p(x)$ is a polynomial of degree less or equal $n-1$, then

$$\min_{\deg(q)\le n-1} \max_{x\in[-1,1]} |x^n - q(x)| = \frac{1}{2^{n-1}}$$

Moreover the q for which the smallest value is attained is exactly $q(x) = x^n - \tilde{T}_n(x)$. For the first few values of n, this is

n	$q_{n-1}(x)$
1	0
2	1/2
3	$3/4\,x$
4	$x^2 - 1/8$

§4.5 # 7* For $n, m \geq 0$ and $n \neq m$, show

$$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}}\, dx = 0$$

This is called the orthogonality relation for the Chebyshev polynomials.

$$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}}\, dx = \int_{-1}^{1} \frac{\cos(n\cos^{-1}(x)) \cos(m\cos^{-1}(x))}{\sqrt{1-x^2}}\, dx$$

Let $x = \cos\theta$ so $dx = \sin\theta\, d\theta$

$$= \int_{\pi}^{0} \frac{(\cos n\theta)(\cos m\theta)}{\sqrt{1-\cos^2\theta}} \sin\theta\, d\theta$$

$$= -\int_{0}^{\pi} (\cos n\theta)(\cos m\theta)\, d\theta$$

Claim that if $m > n$ then.

$$\int_{0}^{\pi} (\cos n\theta)(\cos m\theta)\, d\theta = 0$$

Recal the trigonometric identities

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

Let $a = m\theta$ and $b = n\theta$. Then

§4.5#7* continues...

$$\cos((m+n)x) = \cos mx \cos nx - \sin mx \sin nx$$

$$\cos((m-n)x) = \cos mx \cos nx + \sin mx \sin nx$$

Integrating both equalities from 0 to π and

$$\int_0^\pi \cos((m+n)x)\,dx = \frac{1}{m+n}\sin((m+n)x)\Big|_0^\pi = 0$$

$$\int_0^\pi \cos((m-n)x)\,dx = \frac{1}{m-n}\sin((m+n)x)\Big|_0^\pi = 0$$

yields that

$$\int_0^\pi \cos mx \cos nx\,dx - \int_0^\pi \sin mx \sin nx\,dx = 0$$

and

$$\int_0^\pi \cos mx \cos nx\,dx + \int_0^\pi \sin mx \sin nx\,dx = 0$$

Adding these two equations obtains

$$2\int_0^\pi \cos mx \cos nx\,dx = 0$$

§4.6#4 Most functions do not have $[-1,1]$ as the interval on which they are to be approximated. Suppose $g(t)$ is to be evaluated of $a \le t \le b$. Then define a new function $f(x)$ on $[-1,1]$ by

$$f(x) = g\left(\frac{(b+a) + x(b-a)}{2}\right) \quad \text{to} \quad -1 \le x \le 1.$$

Here

$$t = \frac{1}{2}\left[(b+a) + x(b-a)\right]$$

represents a linear change of variable. We now approximate $f(x)$ on $[-1,1]$.

As a specific example, produce the cubic near min-max approximation for $g(t) = e^t$ with $0 \le t \le 1$. Compare this to the minmax approximation

$$m_3(x) = 0.9994552 + 1.0166023x + 0.4217030x^2 + 0.2799765x^3.$$

In this case $a = 0$ and $b = 1$. Therefore

$$t = \frac{1}{2}(1+x) = \frac{1}{2}x + \frac{1}{2}.$$

The Chebyshev interpolation points on the interval $[0,1]$ are therefore

$$t_n = \frac{1}{2}\cos\left(\frac{2i+1}{2n+2}\pi\right) + \frac{1}{2}$$

In the case that $n = 3$ we obtain

$$t_0 = \frac{1}{2}\cos\left(\frac{\pi}{8}\right) + \frac{1}{2} \approx 0.9619397662$$

$$t_1 = \frac{1}{2}\cos\left(\frac{3\pi}{8}\right) + \frac{1}{2} \approx 0.6913417162$$

$$t_2 = \frac{1}{2}\cos\left(\frac{5\pi}{8}\right) + \frac{1}{2} \approx 0.3086582838$$

$$t_3 = \frac{1}{2}\cos\left(\frac{5\pi}{8}\right) + \frac{1}{2} \approx 0.0380602338$$

§4.6#4

Calculation of the interpolating polynomial was done using Newton's divided difference formula. The Matlab function `divdif.m` written in class

```
function fn=divdif(xn,yn)
    fn=yn;
    n=length(xn);
    for k=1:(n-1)
        for i=n:-1:(1+k)
            fn(i)=(fn(i-1)-fn(i))/(xn(i-k)-xn(i));
        end
    end
```

computes the divided differences and the script `s46p4.m` given as

```
clear all
n=3;
xn=cos((2*[0:n]+1)/(2*n+2)*pi);
tn=1/2*(1+xn);
yn=exp(tn);
fn=divdif(tn,yn);
printf('p:=');
for i=1:n+1
    if i>1
        printf('\n   ');
    end
    printf('%+20.15e',fn(i));
    for j=1:i-1
        printf('*(x%+20.15e)',-tn(j));
        if (mod(j,2)==1) && (j<i-1)
            printf('\n      ');
        end
    end
end
printf(';\n');
```

rescales the node points and prints the interpolating polynomial in an easy to understand form that is suitable to use as input for Maple. The output from the script is

```
p:=+2.616767470796990e+00
   +2.292607583992051e+00*(x-9.619397662556434e-01)
   +9.701904863482995e-01*(x-9.619397662556434e-01)
      *(x-6.913417161825449e-01)
   +2.782396575482458e-01*(x-9.619397662556434e-01)
      *(x-6.913417161825449e-01)*(x-3.086582838174551e-01);
```

§**4.6#4**

The Maple script `s46p4b.mpl` given by

```
restart;
read "s46p4.out";
printf("The near min-max Polynomial is\n");
P:=sort(simplify(p),x,ascending);
printf("The min-max Polynomial is\n");
m3:=0.9994552+1.0166023*x+0.4217030*x^2+0.2799765*x^3;
printf("The difference between them is\n");
P-m3;
```

reads the output from the Matlab program and collects terms in the polynomial for comparison with the min-max polynomial given in the book. The output is

```
p := 0.411417067 + 2.292607583992051 x

     + 0.9701904863482995 (x - 0.9619397662556434) (x - 0.6913417161825449) +

    0.2782396575482458 (x - 0.9619397662556434) (x - 0.6913417161825449)

    (x - 0.3086582838174551)

The near min-max Polynomial is
                                                              2                   3
     P := 0.9995086155 + 1.015632510 x + 0.4243010377 x  + 0.2782396575 x

The min-max Polynomial is
                                                         2                3
          m3 := 0.9994552 + 1.0166023 x + 0.4217030 x  + 0.2799765 x

The difference between them is
                                                         2                   3
        0.0000534155 - 0.000969790 x + 0.0025980377 x  - 0.0017368425 x
```

The coefficents for the two polynomials are almost the same. In particular, they differ by no more than about 1 percent.

```
> # S4.7p3 Find least squares approximations of degree less or equal 4.
  restart;
```

```
> n:=4;
  P[0]:=1;
  for i from 1 to n
  do
      P[i]:=unapply(1/i!/2^i*diff((x^2-1)^i,x$i),x);
  end;
```

$$n := 4$$

$$P_0 := 1$$

$$P_1 := x \to x$$

$$P_2 := x \to \frac{3}{2}\,x^2 - \frac{1}{2}$$

$$P_3 := x \to x^3 + \frac{3}{2}\,(x^2 - 1)\,x$$

$$P_4 := x \to x^4 + 3\,(x^2 - 1)\,x^2 + \frac{3}{8}\,(x^2 - 1)^2$$

```
> dp:=(f,g)->int(f(x)*g(x),x=-1..1);
```

$$dp := (f, g) \to \int_{-1}^{1} f(x)\, g(x)\, dx$$

```
> lspoly:=f->sum(dp(f,P[j])/dp(P[j],P[j])*P[j](x),j=0..n);
```

$$lspoly := f \to \sum_{j=0}^{n} \frac{dp(f, P_j)\, P_j(x)}{dp(P_j, P_j)}$$

```
> doit:=proc(f)
      local p;
      printf("We are considering the function\n");
      print(f(x));
      printf("The least squares poly of degree %g or less is\n",n);
      p:=sort(collect(lspoly(f),x),x,ascending);
      print(p);
      printf("or approximately\n");
      print(sort(evalf(p),x,ascending));
      printf("A graph of the errors on [-1,1] is\n");
      plot(f(x)-p,x=-1..1);
  end:
```

```
>
```

```
> # S4.7p3 part a.
  doit(x->sin(Pi*x));
```

We are considering the function

$$\sin(\pi\, x)$$

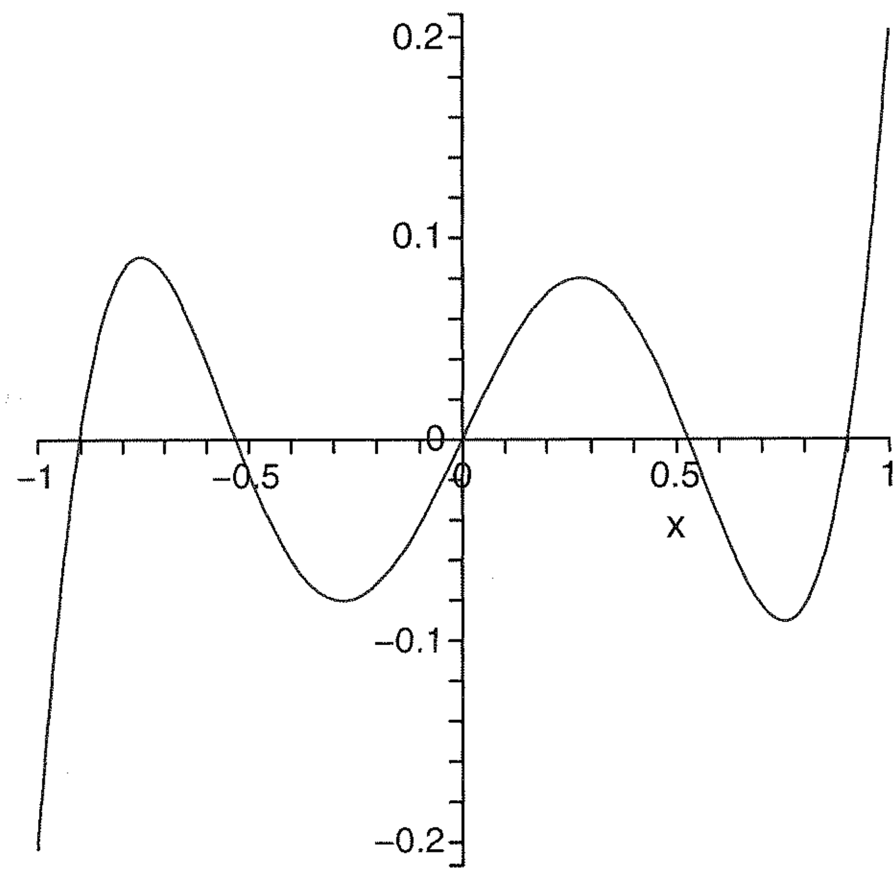
The least squares poly of degree 4 or less is

$$\left(\frac{3}{\pi} - \frac{21\,(-15+\pi^2)}{2\,\pi^3}\right) x + \frac{35\,(-15+\pi^2)\,x^3}{2\,\pi^3}$$

or approximately

$$2.692292525\,x - 2.895604778\,x^3$$

A graph of the errors on [-1,1] is




```
>
```

```
> # S4.7p3 part b.
  doit(x->log(1+x^2));
```

We are considering the function

$$\ln(x^2+1)$$

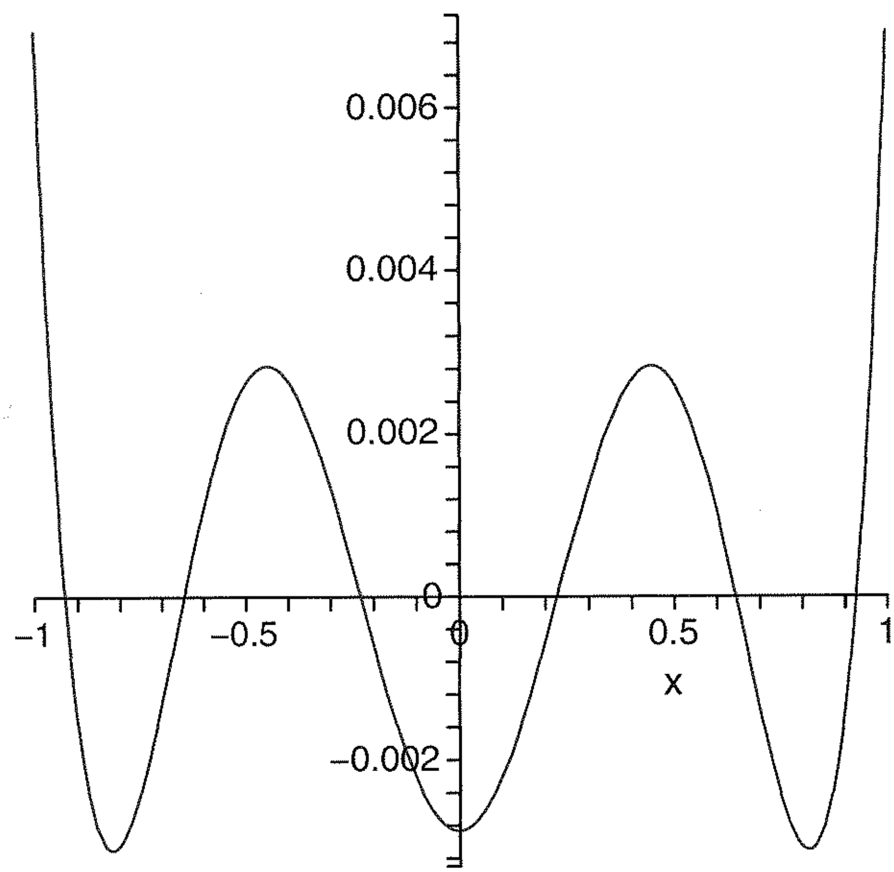
The least squares poly of degree 4 or less is

$$\frac{191}{32}\pi + \ln(2) - \frac{2333}{120} + \left(-\frac{735}{16}\pi + \frac{581}{4}\right)x^2 + \left(\frac{1575}{32}\pi - \frac{1239}{8}\right)x^4$$

or approximately

$$0.00286166 + 0.9330875\,x^2 - 0.2497366\,x^4$$

A graph of the errors on [-1,1] is




```
>
```

```
> # S4.7p3 part c.
  doit(x->arctan(x));
```

We are considering the function

$$\arctan(x)$$

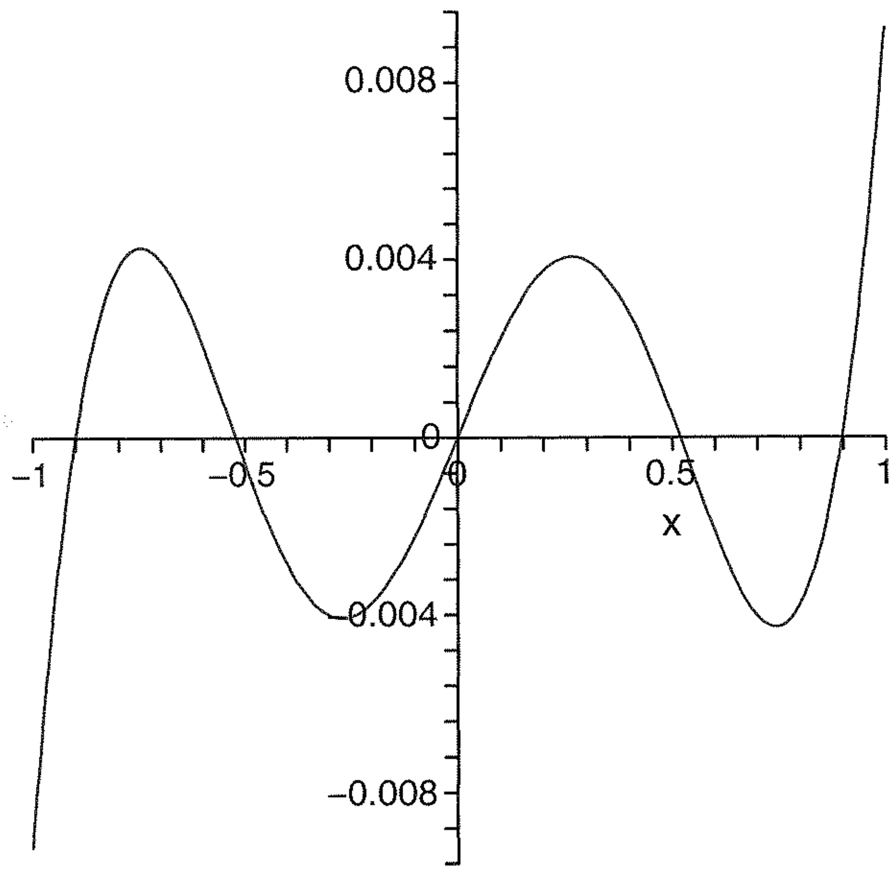
The least squares poly of degree 4 or less is

$$\left(\frac{75}{16}\pi - \frac{55}{4}\right)x + \left(-\frac{105}{16}\pi + \frac{245}{12}\right)x^3$$

or approximately

$$0.97621557\,x - 0.20003512\,x^3$$

A graph of the errors on [-1,1] is




```
>
```

```
> # S4.7p3 part d.
  doit(x->exp(x));
```

We are considering the function

$$e^{x}$$

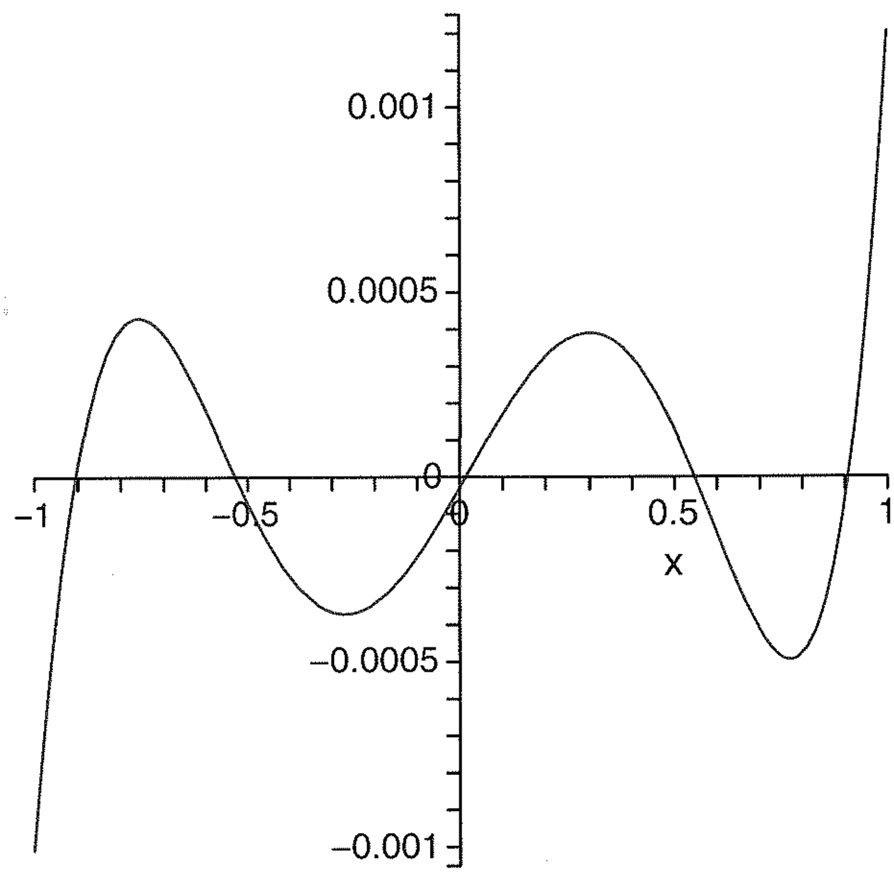
The least squares poly of degree 4 or less is

$$-\frac{3525}{8}e^{(-1)}+60\,e+\left(-\frac{765}{4}e^{(-1)}+\frac{105}{4}e\right)x+\left(\frac{8925}{2}e^{(-1)}-\frac{2415}{4}e\right)x^2+\left(\frac{1295}{4}e^{(-1)}-\frac{175}{4}e\right)x^3+\left(-\frac{41895}{8}e^{(-1)}+\frac{2835}{4}e\right)x^4$$

or approximately

$$1.0000309+0.99795485\,x+0.499352\,x^2+0.1761391\,x^3+0.043597\,x^4$$

A graph of the errors on [-1,1] is




```
>
```