1. Consider the divided differences for $f(x) = (\pi - x)\log(x)$ given by

x_n	$f(x_n)$	$f(x_n, x_{n+1})$	$f(x_n, \ldots, x_{n+2})$	$f(x_n, \ldots, x_{n+3})$	$f(x_n, \ldots, x_{n+4})$
1.5	0.66561	0.25137	−0.65818	0.13355	−0.02898
2.0	0.79129	−0.40681	?	0.07560	
2.5	0.58789	−0.86466	−0.34445		
3.0	0.15556	−1.20911			
3.5	−0.44900				

(i) Find the missing divided difference in the above table.

(ii) Find the interpolating polynomial passing through the points

$$(1.5, 0.66561),\ (2, 0.79129),\ (2.5, 0.58789),\ (3, 0.15556)$$

using Newton's divided difference formula.

2. Fill in the blanks in the following statement of Theorem 4.2.1 on the error in polynomial interpolation.

Theorem 4.2.1. Let $n \geq 0$, let $f(x)$ have $n+1$ continuous derivatives on $[a, b]$, and let $x_0, x_1, \ldots, x_n$ be distinct node points in $[a, b]$. Let $p_n(x)$ be the unique interpolating polynomial of degree less than or equal n passing through the points $(x_k, f(x_k))$ for $k = 0, 1, \ldots, n$. Then

$$f(x) - p_n(x) = \boxed{}$$

for $a \leq x \leq b$, where c_x is an unknown point between the minimum and maximum of $x_0, x_1, \ldots, x_n$, and x.

3. The first three Chebyshev polynomials are $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$. Define the Chebyshev polynomial of degree n, state what important properties these polynomials have and what they are used for.

4. The triple recurrence relation for the Chebyshev polynomials is

(A) $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

(B) $T_{n+1}(x) = 2xT_n(x) + T_{n-1}(x)$.

(C) $T_{n+1}(x) = T_n(x) - 2xT_{n-1}(x)$.

(D) $T_{n+1}(x) = T_n(x) + 2xT_{n-1}(x)$.

(E) none of these.

5. Define the Legendre polynomial $P_n(x)$ of degree n.

(A) $P_n(x) = \cos(n \cos^{-1}(x))$.

(B) $P_n(x) = \cos^{-1}(n \cos(x))$.

(C) $P_n(x) = \dfrac{1}{n!2^n} \dfrac{d^n}{dx^n} \big[(x^2+1)^n\big]$.

(D) $P_n(x) = \dfrac{1}{n!2^n} \dfrac{d^n}{dx^n} \big[(x^2-1)^n\big]$.

(E) none of these.

6. Let $P_n(x)$ and $P_m(x)$ be Legendre polynomials where $n \neq m$. Then

(A) $\displaystyle\int_{-1}^{1} P_n(x)P_m(x)\,dx = 1$

(B) $\displaystyle\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$

(C) $\displaystyle\int_{-1}^{1} \big(P_n(x)+P_m(x)\big)^2 dx = 1$

(D) $\displaystyle\int_{-1}^{1} \big(P_n(x)+P_m(x)\big)^2 dx = 0$

(E) none of these.

7. Finish the following statement:

Given an integrable function $f(x)$ defined on the interval $[-1,1]$ let $p(x)$ be the polynomial of degree less than or equal n such that the integral

$$\int_{-1}^{1} \big|f(x)-p(x)\big|^2 dx$$

is minimized. Then

$$p(x) = \sum_{j=0}^{n} \beta_j P_j(x)$$

where $P_j(x)$ are the Legendre polynomials and β_j is given by ...

8. Is

$$s(x) = \begin{cases} (x-1)^3 & \text{for } 0 \le x \le 1 \\ 2(x-1)^3 & \text{for } 1 \le x \le 2 \end{cases}$$

a cubic spline function on the interval $0 \le x \le 2$? Show your work and explain.

9. Write pseudocode to implement the trapezoid method for finding

$$\int_a^b f(x)\,dx.$$

The inputs for your code should be f, a, b and n where n is the number of equally spaced subdivisions of the interval $[a,b]$ used in the method.