Three Stage Runge–Kutta Methods

1. Let $y(t)$ be the exact solution to the initial value problem

$$y' = f(t, y), \qquad y(0) = y_0.$$

Suppose $y(t)$ and its derivatives are bounded for all $t \in [0, T]$. Given $n \in \mathbf{N}$ define $t_i = ih$ where $h = T/n$. Let y_i be the approximation of $y(t_i)$ given by the Shu–Osher TVD Runge–Kutta scheme

$$\begin{array}{c|ccc} 0 & & & \\ 1 & 1 & & \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array}.$$

Prove $y(t_1) = y_1 + \mathcal{O}(h^4)$ to show this a third order method.

2. Use the Shu–Osher TVD Runge–Kutta scheme to approximate the solution to

$$y' = y^2 \cos t, \qquad y(0) = 0.8$$

on the interval $[0, 8]$ for $n = 50$. Graph your approximation.

3. Verify that the exact solution to this equation is

$$y(t) = \frac{y_0}{1 - y_0 \sin t}.$$

4. Let y_n be the approximation of $y(8)$ obtained by the Shu–Osher TVD Runge–Kutta scheme using n equal steps of size $h = 8/n$. Graph $\log |y_n - y(8)|$ versus $\log h$ to verify the order of convergence found in part 1 numerically.

5. [Extra Credit and Math/CS 666] The classical Runge–Kutta scheme and the Nystrom Runge–Kutta schemes are given by

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} \qquad \text{and} \qquad \begin{array}{c|ccc} 0 & & & \\ \frac{2}{3} & \frac{2}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{array}$$

respectively. Let z_n be the approximation of $y(8)$ obtained from the classical RK scheme and w_n be obtained from the Nystrom RK scheme using n equal steps of size $h = 8/n$. Compare $\log |z_n - y(8)|$ and $\log |w_n - y(8)|$ to the values of $\log |y_n - y(8)|$ for $n = 50$ and $n = 100$. Which scheme is preferrable when solving the equation in part 2?